Mathematics 264 Assignment 5

Due on Tuesday November 7, 2006 in class

(1) (a) Show that

$$\iint_{(S)} \textbf{curl } \mathbf{F} \cdot \hat{\mathbf{N}} \mathrm{d}S = 0,$$

where $\mathbf{F}$ is an arbitrary smooth vector field, (S) is the boundary of a simple connected domain (D).

(b) If $\mathbf{F}$ is smooth vector field on the domain (D), show that

$$\iiint_{(D)} \phi \textbf{div } \mathbf{F} \mathrm{d}V + \iiint_{(D)} \nabla\phi \cdot \mathbf{F} \mathrm{d}V = \iint_{(S)} \phi \mathbf{F} \cdot \hat{\mathbf{N}} \mathrm{d}S,$$

where (S) is the boundary of (D).

(2) (a) Evaluate

$$\iint_{(S)} \nabla \times \mathbf{F} \cdot \hat{\mathbf{N}} \mathrm{d}S,$$

where (S) is the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ with outward normal, and

$$\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}.$$

(b) Let $(\mathcal{C}_1)$ be the straight line joining $(-1, 0, 0)$ to $(1, 0, 0)$, and let $(\mathcal{C}_2)$ be the semicircle $x^2 + y^2 = 1, z = 0, y \geq 0$. Let (S) is a smooth surface joining $(\mathcal{C}_1)$ to $(\mathcal{C}_2)$ having upward normal, and

$$\mathbf{F} = (\alpha x^2 - z)\mathbf{i} + (xy + y^3 + z)\mathbf{j} + \beta y^2(z + 1)\mathbf{k}.$$

Find the value of α and β for which

$$I = \iint_{(S)} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

is independent of the choice of (S), and find the value of I for these values of α and β.

(3) (a) (i) A string has its ends fixed at $x = 0$, and $x = L$. It is displaced a distance h at its midpoint and then released. Formulate a boundary value problem for the displacement $y(x, t)$ of any point x of the string at time t, (ii) Solve the above problem to receive a bonus point.

(b) Classify each of the following equations as elliptic, hyperbolic or parabolic:

$$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} = 4,$$

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial u}{\partial y^2} = x + 3y,$$

$$(M^2 - 1)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y^2} = 0, (M > 0).$$

(4) (a) Show that $v(x, y) = xF(2x + y)$ is a general solution of the equation

$$x\frac{\partial v}{\partial x} - 2x\frac{\partial v}{\partial y} = v.$$

(b) Show that $u = F(3x + y) + G(3x - y)$ is the general solution of the equation.

$$\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial y^2} = 0,$$

where F, G are arbitrary functions.