MATH 265, Advanced Calculus
Solutions to Assignment 2

1. We use the obvious substitution: $u = x^2 - y^2$, $v = xy$. D is then described simply by $1 \le u \le 3$, $2 \le v \le 3$. $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}$ with determinant $2x^2 + 2y^2$; the Jacobian we need is then $\frac{1}{2x^2+2y^2}$. Fortunately, we don't need to solve for x and y in terms of u and v, since the integrand becomes $J \cdot (x^4 - y^4) = \frac{x^2-y^2}{2} = \frac{u}{2}$. The double integral becomes
$$\int_2^3 \int_1^3 \frac{u}{2} du dv = \int_2^3 2 dv = 2.$$

2. Let $u = x + y$, $v = 2x - y$, $w = y + z$. D is then given by $1 \le u \le 4$, $-1 \le v \le 3$, $-1 \le w \le 2$. $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ with determinant -3; the Jacobian we need is then $-\frac{1}{3}$, but we also change sign before putting it into the integral. Also, $x = \frac{u+v}{3}$ — we don't need to solve for y or z. The integral becomes
$$\int_{-1}^2 \int_{-1}^3 \int_1^4 \frac{1}{3} \frac{u+v}{3} du dv dw = 14.$$

3. We describe the region using cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. We have $0 \le z \le \frac{a}{\sqrt{1+b^2}}$; $bz \le r \le \sqrt{a^2 - z^2}$; $0 \le \theta \le 2\pi$. (We are assuming WLOG that a and b are positive. We get $0 \le z$ as we are above the xy-plane. The upper limit for z come from equating $r^2 = x^2 + y^2 = a^2 - z^2 = b^2 z^2$.) The Jacobian is just r, so the volume is given by
$$\int_0^{2\pi} d\theta \int_0^{\frac{a}{\sqrt{1+b^2}}} dz \int_{bz}^{\sqrt{a^2-z^2}} r dr = \frac{2a^3\pi}{3\sqrt{1+b^2}}.$$
The numerator for the z-coordinate of the centroid is
$$\int_0^{2\pi} d\theta \int_0^{\frac{a}{\sqrt{1+b^2}}} dz \int_{bz}^{\sqrt{a^2-z^2}} rz dr = \frac{a^4\pi}{4(1+b^2)}$$
and so the centroid is $(0, 0, \frac{3a}{8\sqrt{1+b^2}})$. (The centroid is on the z-axis of evil because clearly the region is symmetric about that line; so we are spared calculating the x- and y-coordinates. Note also that this point is not inside the region.)

4. Again, we need not compute the x-coordinate or the y-coordinate; by symmetry they are both zero. We find the numerator and denominator of the z-coordinate. The denominator is the surface area, given by

$\int\int_D \sqrt{1+z_x^2+z_y^2}dxdy$, where $z_x = -2x$, $z_y = -2y$ and D is the disc $0 \le x^2+y^2 \le 4$. The integrand is $\sqrt{1+4(x^2+y^2)}$, so polar coordinates are definitely called for; the integral becomes

$$\int_0^{2\pi} d\theta \int_0^2 r\sqrt{1+4r^2}dr.$$

Substituting $u = 1+4r^2$, $du = 8rdr$, we integrate $\frac{1}{8}\int \sqrt{u}du = \frac{1}{12}u^{\frac{3}{2}} = \frac{1}{12}(1+4r^2)^{\frac{3}{2}}$. This is evaluated at 2 and 0, and multiplied by 2π to give a surface area of $A = \frac{\pi}{6}(17^{\frac{3}{2}}-1)$.

The numerator is $\int\int_D z\sqrt{1+z_x^2+z_y^2}dxdy$, or in polar coordinates (using $z = 4-x^2-y^2 = 4-r^2$),

$$\int_0^{2\pi} d\theta \int_0^2 r(4-r^2)\sqrt{1+4r^2}dr.$$

$u = 1+4r^2$, $du = 8rdr$, $4-r^2 = \frac{17-u}{4}$ leaves us to integrate $\frac{1}{32}\int(17u^{\frac{1}{2}} - u^{\frac{3}{2}})du = \frac{1}{32}(\frac{34}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}})$. We substitute back $u = 1+4r^2$ and evaluate from 0 to 2, and then multiply by 2π, to get a numerator of $\frac{\pi}{16}(\frac{34}{3}17^{\frac{3}{2}} - \frac{2}{5}17^{\frac{5}{2}} - \frac{164}{15}) = \frac{\pi}{60}(17^{\frac{5}{2}} - 41)$. Call this B; the z-coordinate is $\frac{B}{A}$.

The masochistic among us might actually care to simplify this expression for this number; I got $\frac{10430-51\sqrt{17}}{6140}$.

5. We want $\phi(x,y)$ such that $\phi_x = y$ and $\phi_y = x$. Trivially, $\phi(x,y) = xy$ satisfies these conditions, so this is a fine potential function. The equipotential curves are defined by $xy = C$ for various constants C; this equation defines a hyperbola if $C \ne 0$, and the coordinates axes if $C = 0$.

   For the field lines, we solve $\frac{dx}{y} = \frac{dy}{x}$ or $xdx = ydy$, so $x^2 = y^2 + K$ for some constant K. If $K \ne 0$, the graph of this is a hyperbola again; if $K = 0$ we get the pair of lines $y = \pm x$.

6. Integrating $-\frac{y^2+2z^2}{x^2}$ with respect to x (holding y and z constant) gives $\phi(x,y,z) = \frac{y^2+2z^2}{x} + C(y,z)$. We can choose $C(y,z) = 0$ — usually we can't — and we have $\phi_y = \frac{2y}{x}$ and $\phi_z = \frac{4z}{x}$. A potential potential function is $\phi(x,y,z) = \frac{y^2+2z^2}{x}$. Setting this equal to C and simplifying gives $y^2+2z^2 = Cx$. The equipotential curves are paraboloids with vertex at the origin and x-axis as the axis of symmetry.

   For the field lines, we solve

$$\frac{dx}{-\frac{y^2+2z^2}{x}} = \frac{dy}{\frac{2y}{x}} = \frac{dz}{\frac{4z}{x}}.$$

The second equation simplifies to $2\frac{dy}{y} = \frac{dz}{z}$, so $\ln|z| = 2\ln|y| + C$, and $z = Ky^2$, a parabolic cylinder for $K \neq 0$. The first equation now becomes $-\frac{dx}{y^2+2K^2y^4} = \frac{dy}{2y}$, or $-dx = (\frac{1}{2}y + K^2y^3)dy$, so $x = -(\frac{1}{4}y^2 + \frac{1}{2}K^2y^4 + L)$ (a quartic curve in the xy-plane stretched to a cylinder in 3-space). The field lines are the intersections of these cylinders with the previous ones, naturally parametrized with $y = t$. For any fixed constants K and L we have $x = -(\frac{1}{4}t^2 + \frac{1}{2}K^2t^4 + L)$, $y = t$, $z = Kt^2$.

7. We parametrize by $x = \cos\theta$, $z = \sin\theta$, $y = \cos^2\theta$, $0 \leq \theta \leq 2\pi$. Then $dx = -\sin\theta d\theta$, $dy = \cos\theta d\theta$, $dy = -2\cos\theta\sin\theta d\theta$. $ds = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2 + (-2\cos\theta\sin\theta)^2}d\theta = \sqrt{1 + 4\cos^\theta\sin^2\theta}d\theta$. Of course the expression under the surd is just $1 + 4x^2z^2$, so we integrate $\int_0^{2\pi}(1 + 4\cos^2\theta\sin^2\theta)d\theta = \int_0^{2\pi}(\frac{3}{2} - \frac{\cos 4\theta}{2})d\theta = 3\pi$.

8. Here, $ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt = \sqrt{\sin^2 t + \cos^2 t + 1}dt = \sqrt{2}dt$. For the mass, $m = \int_C \delta ds = \int_0^\pi t\sqrt{2}dt = \frac{\pi^2\sqrt{2}}{2}$. $\int_C x\delta ds = \int_0^\pi t\cos t\sqrt{2}dt = -2\sqrt{2}$, $\int_C y\delta ds = \int_0^\pi t\sin t\sqrt{2}dt = \pi\sqrt{2}$ and $\int_C z\delta ds = \int_0^\pi t^2\sqrt{2}dt = \frac{\pi^3\sqrt{2}}{3}$. So the centre of mass is $(-\frac{4}{\pi^2}, \frac{2}{\pi}, \frac{2\pi}{3})$.