MATH 265, Advanced Calculus
Solutions to Assignment 5

1. For this $\mathbf{F}$, $\nabla \cdot \mathbf{F} = \mathbf{2x} + \mathbf{2y} - \mathbf{2}(\mathbf{x} + \mathbf{y}) = \mathbf{0}$, so $\mathbf{F}$ is solenoidal. We take $G_3 = 0$ and we solve $-(G_2)_z = x^2 + yz$, $(\hat{G}_1)_z = y^2 + zx$. Say $G_2 = -x^2 z - \frac{1}{2}yz^2$ and $\hat{G}_1 = y^2 z + \frac{1}{2}xz^2$. Then for $\mathbf{G}$ to be a vector potential, we need $(G_2)_x - (G_1)_y = -2zx - 2zy$; this is already true with $\hat{G}_1 = G_1$, so we don't need to make the usual adjustment.
$\mathbf{G} = (y^2 z + \frac{1}{2}xz^2)\mathbf{i} + (-x^2 z - \frac{1}{2}yz^2)\mathbf{j} + 0\mathbf{k}$ is a vector potential for $\mathbf{F}$.

2. We do (b) first. Since $\rho\mathbf{a} = \nabla p$, $\nabla \times (\rho\mathbf{a}) = \nabla \times (\nabla p) = \mathbf{0}$. Also, $\mathbf{0} = \nabla \times (\rho\mathbf{a}) = \rho(\nabla \times \mathbf{a}) + (\nabla\rho \times \mathbf{a})$, so $\rho(\nabla \times \mathbf{a}) = -(\nabla\rho \times \mathbf{a}) = \mathbf{a} \times \nabla\rho$, which is (c).

   Finally, $\mathbf{a} \cdot \mathbf{0} = \mathbf{a} \cdot [\rho(\nabla \times \mathbf{a}) + (\nabla\rho \times \mathbf{a})] = \mathbf{a} \cdot [\rho(\nabla \times \mathbf{a})] + \mathbf{a} \cdot (\nabla\rho \times \mathbf{a})$.

   $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{0}$ for any vector field $\mathbf{b}$, so this implies $\rho(\mathbf{a} \cdot (\nabla \times \mathbf{a})) = \mathbf{0}$. Cancelling ρ finishes (a) and the problem.

3. (a) This one is probably just as easily done straight, but let's use Green's Theorem anyway. The line integral is $\int\int_R ((F_2)_x - (F_2)_y)dA = \int\int_R (x + y)dA$, where R is the interior of the triangle. We break it into two parts, and evaluate

$$\int_0^1 \int_0^x (x + y)dydx + \int_1^2 \int_0^{2-x} (x + y)dydx =$$

$$\int_0^1 (xy + \frac{1}{2}y^2]_0^x)dx + \int_1^2 (xy + \frac{1}{2}y^2]_0^{2-x})dx = \frac{1}{2}x^3]_0^1 + \int_1^2 (2 - \frac{1}{2}x^2)dx = \frac{1}{2} + 2 - \frac{7}{6} = \frac{4}{3}.$$

   (b) This time, Green definitely makes life easier. Since the boundary curve is traversed in the negative (clockwise) direction, we evaluate $\int\int_R ((F_1)_y - (F_2)_x)dA = \int\int_R (x^2 + y^2)dA$. Obviously, polar coordinates are called for; we have $0 \le r \le 3$, $0 \le \theta \le \pi$, so the integral is $\int_0^\pi \int_0^3 r^3 drd\theta = \frac{81}{4}\pi$.

4. The plane containing the triangle is defined by $z = -\frac{c}{a}x - \frac{c}{b}y + c$, with normal $\vec{N}_{x,y} = \frac{c}{a}\mathbf{i} + \frac{c}{b}\mathbf{j} + \mathbf{k}$ pointing upwards. $\nabla\phi = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$.
x and y come from the region given by $0 \le x \le a$, $0 \le y \le b - \frac{b}{a}x$ and we integrate $\nabla\phi \cdot \vec{N}_{x,y}$ over this region.

$$\int_0^a \int_0^{b - \frac{b}{a}x} \frac{c}{a}y + \frac{c}{b}x + 2(-\frac{c}{a}x - \frac{c}{b}y + c)dydx =$$

$$\int_0^a [\frac{b^2c + 2abc}{2a} + \frac{a^2c - b^2c - 2abc}{a^2}x + \frac{b^2c + 2abc - 2a^2c}{2a^3}x^2] = \frac{a^2c + b^2c + 2abc}{6}.$$

This is correct, but not a very sensible way to go about it. Loveys, get a clue! The divergence theorem tells us that

$$\int\int\int_V \nabla\cdot(\nabla\phi)dV = \int\int_S (\nabla\phi)\cdot\hat{N}dS$$

where S is the boundary of V; take V as the tetrahedron with vertices $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ and $(0,0,0)$. $\nabla\cdot(\nabla\phi) = 2$ so the triple integral is twice the volume of the tetrahedron — it's $2(\frac{abc}{6})$. S consists of four triangles; the one we are interested in and one on each coordinate plane. The surface integrals on the coordinate planes are easy to compute: On the bottom (where $z = 0$, the normal is $-\mathbf{k}$, so we are integrating 0 over a triangle, and that integral is 0.

On the yz-plane we have $x = 0$ and the normal is $-\mathbf{i}$. We integrate $\nabla\phi\cdot(-\mathbf{i}) = -y$ over the triangle $0 \le y \le b$, $0 \le z \le c - \frac{c}{b}y$ and get $\int_0^b \int_0^{c-\frac{c}{b}y} -y dz dy = \int_0^b (-cy + \frac{c}{b}y^2)dy = -\frac{1}{6}b^2c$. Similarly, the integral on the piece on the xz plane is $-\frac{a^2c}{6}$. Thus $\frac{2abc}{6} = \int\int_T (\nabla\phi\cdot\hat{N})dS + 0 - \frac{b^2c}{6} - \frac{a^2c}{6}$ where T is the tiangle we are really after. The answer $\frac{a^2c+b^2c+2abc}{6}$ jibes with what we had before — reassuring, that.

5. The surface is naturally parametrized by: $x = x$, $y = \cos\theta$, $z = \sin\theta$, $0 \le x \le 1$, $0 \le \theta \le \frac{\pi}{2}$. $\vec{N}_{\theta,x} = \cos\theta\mathbf{j} + \sin\theta\mathbf{k}$ points away from the origin. $(3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k})\cdot\vec{N} = -x\cos\theta - \cos\theta\sin\theta$, so we want

$$\int_0^1 \int_0^{\frac{\pi}{2}} -x\cos\theta - \frac{\sin 2\theta}{2} d\theta dx = -1.$$

6. $\nabla\cdot(\nabla\times(\cos x, y^3\sin xy, z)) = 0$ throughout the region, so by the divergence theorem, the double integral for this part is zero. Also by Da Big Guy's theorem (really), the flux integral of $\dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ is 4π over any closed surface — such as this ellipsoid — that has the origin in its interior. The answer is 4π. (You should be able to provide the details as in Example 4 on page 970 of Adams).