MATH 264B Advanced Calculus, Winter 2006

Assignment 3, due in class on Thursday February 16, 2006, Solution **outlines**.

1. A parametrization for S is given by $\mathbf{X}(u,v) = (u\cos v, u\sin v, u)$ where $0 < u < R$ and $0 < v < 2\pi$. This gives $||\mathbf{X}_u \times \mathbf{X}_v|| = \sqrt{2}u$, and the surface integral to be computed reduces to
$$\int_0^R \int_0^{2\pi} \sqrt{2}u \, dvdu = \pi\sqrt{2}R^2.$$

2. A parametrization for S is given by
$$\mathbf{X}(u,v) = (2,0,0) + u[(1,0,1) - (2,0,0)] + v[(2,1,0) - (2,0,0)],$$
that is
$$\mathbf{X}(u,v) = (2-u, v, u)$$
where $0 < u < 1$ and $0 < v < 1$. We have $||\mathbf{X}_u \times \mathbf{X}_v|| = \sqrt{2}$, and the surface integral to be computed reduces to
$$\sqrt{2}\int_0^1 \int_0^1 (2-u)\, dudv = 3/\sqrt{2}.$$

3. A parametrization for S is given by $\mathbf{X}(u,v) = (u\cos v, u\sin v, u)$ where $1 < u < 2$ and $0 < v < 2\pi$. With this parametrization, the normal
$$\mathbf{X}_u \times \mathbf{X}_v = (-u\cos v, -u\sin v, u)$$
is inward pointing. The surface integral to be computed reduces to
$$\int_1^2 \int_0^{2\pi} (u^2\cos v, u^2\sin v, u^2).(u\cos v, u\sin v, -u)\, dvdu$$
that is
$$\int_1^2 \int_0^{2\pi} (u^3\cos^3 v + u^3\sin^3 v - 1) dvdu = -15\,\pi/2.$$

4. We decompose our surface integral as the sum
$$\iint_{S_+} \mathbf{F}.\mathbf{N}\, dS + \iint_{S_-} \mathbf{F}.\mathbf{N}\, dS,$$
where S_+ denotes the upper unit hemisphere and S_- denotes the unit equatorial disk (the bottom lid). The outward pointing unit normal $\mathbf{N}$ at a point (x,y,z) of the

upper unit hemisphere S_+ is given by the position vector (x, y, z) itself. The outward flux of $\mathbf{F} = (2x, 2y, 2z)$ through S_+ is thus given by

$$\iint_{S_+} \mathbf{F}.\mathbf{N}\, dS = \iint_{S_+} (2x^2 + 2y^2 + 2z^2)\, dS.$$

But we have $2x^2 + 2y^2 + 2z^2 = 2$ at every point of S_+, so that

$$\iint_{S_+} \mathbf{F}.\mathbf{N}\, dS = 2\, Area(S_+) = 4\pi.$$

The outward pointing unit normal $\mathbf{N}$ at a point $(x, y, 0)$ of the disk is given by $(0, 0, -1)$, which is orthogonal to $\mathbf{F} = (2x, 2y, 0)$ at that point, so that

$$\iint_{S_-} \mathbf{F}.\mathbf{N}\, dS = 0.$$

The total flux is therefore equal to 4π.

5. A parametrization for S is given by $\mathbf{X}(\theta, z) = (\cos\theta, \sin\theta, z)$ where $0 < \theta < 2\pi$ and $0 < z < 1$. We have

$$\mathbf{X}_\theta \times \mathbf{X}_z = (\cos\theta, \sin\theta, 0),$$

which is outward pointing. The total flux is thus given by

$$\int_0^{2\pi} \int_0^1 (\cos\theta, \sin\theta, -\sin\theta).(\cos\theta, \sin\theta, 0)\, d\theta dz = 2\pi.$$