MATH 264B Advanced Calculus, Winter 2006

Assignment 4. Solution outlines.

1. Simple computations produce

$$div\ \mathbf{F} = 0$$

$$curl\ \mathbf{F} = \frac{1}{2}\left(\frac{\sqrt{x}}{\sqrt{y}} - \frac{x}{z\sqrt{z}}\right)\mathbf{i} - \frac{1}{2}\left(\frac{\sqrt{y}}{\sqrt{x}} + \frac{y}{z\sqrt{z}}\right)\mathbf{j} - \frac{2}{\sqrt{z}}\mathbf{k}.$$

2. Take $\mathbf{F}(x, y, z) = \frac{1}{2}(-y\mathbf{i} + x\mathbf{j})$. Hence

$$Area = \oint_{\mathcal{C}_1} \mathbf{F}\ d\mathbf{r} + \oint_{\mathcal{C}_2} \mathbf{F}\ d\mathbf{r},$$

where $\mathcal{C}_1$ is oriented from left to right (from $-\pi$ to 0) and $\mathcal{C}_2$ is oriented as shown on the picture. Obviously,

$$\oint_{\mathcal{C}_1} \mathbf{F}\ d\mathbf{r} = \int_{-\pi}^{0} 0\ dx = 0.$$

The parametrization of $\mathcal{C}_2$ is $\mathbf{r}(t) = t\cos t\mathbf{i} + 2t\sin t\mathbf{j}$, $0 \le t \le \pi$, so $\mathbf{r_1'}(t) = (\cos t - t\sin t)\mathbf{i} + 2(\sin t + t\cos t)\mathbf{j}$ and $\mathbf{F}\cdot d\mathbf{r} = t^2$. Thus,

$$\oint_{\mathcal{C}_2} \mathbf{F}\ d\mathbf{r} = \int_0^{\pi} t^2\ dt = \frac{\pi^3}{3},$$

and

$$Area = 0 + \frac{\pi^3}{3} = \frac{\pi^3}{3}.$$

3. Let $\mathbf{F}(x, y, z) = (x + y)^2\mathbf{i} - (x^2 + y^2)\mathbf{j}$. Then $curl\ \mathbf{F} = -(4x + 2y)\mathbf{k}$, and

$$\oint_{\mathcal{C}} (x + y)^2\ dx - (x^2 + y^2)\ dy = -\int_0^2 dy \int_{y+1}^{3} (4x + 2y)\ dx = -\frac{64}{3}.$$

4. Let $\mathbf{F}(x, y, z) = (xy + y^2)\mathbf{i} - (x^2 - y)\mathbf{j}$. Then $curl\ \mathbf{F} = -(3x + 2y)\mathbf{k}$, and

$$\oint_{\mathcal{C}} (xy + y^2)\ dx + (x^2 - y)\ dy = \int\int_D (x - 2y)\ dA$$

$$= \int_1^3 dr \int_0^{2\pi} (r\cos\theta - 2r\sin\theta)r\ d\theta = 0.$$

5. Observe that if we denote by $\mathcal{C}$ the cardioid and by D the region bounded by the cardioid, then

$$\oint_{\mathcal{C}} \mathbf{F}\ d\mathbf{r} = Area(D).$$

The cardioid curve is symmetric about the x-axis, so it is enough to compute the area above the x-axis and then double the result. Thus we have

$$Area(D) = 2\left(\frac{1}{2}\int_0^{\pi} (1 + \cos\theta)^2 d\theta\right) = \frac{3\pi}{2}.$$