McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 265

ADVANCED CALCULUS

Examiner: Professor W. Jonsson
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Date: Thursday, December 12, 2002
Time: 9:00 A.M. - 12:00 P.M.

INSTRUCTIONS

Attempt all questions.
Calculators are not permitted.
The questions are not necessarily of equal weight.

This exam comprises the cover and 1 page of 7 questions.

1. Consider the vector field

$$\mathbf{F} = \nabla\Big(\frac{1}{\sqrt{(x+1)^2 + y^2 + (z-3)^2}} + e^x \cos y\Big).$$

   (a) Compute the flux of $\mathbf{F}$ across the surface of the cube centered at $(0,1,0)$, with edges of length 8, oriented with the outward-pointing normal.

   (b) Same question as in (a), but with the center of the cube located at $(17, 21, 55)$.

2. Consider the equation

$$z^3 - xz - y = 0.$$

   (a) Give a sufficient condition for being able to solve for z as a differentiable function of (x, y) near a point (x_0, y_0, z_0).

   (b) Show by implicit differentiation that near any point (x_0, y_0, z_0) satisfying the condition obtained in (a), the function $z(x, y)$ satifies

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{3z^2 + x}{(3z^2 - x)^3}.$$

3. Let S be the subset of the surface of the sphere $x^2 + y^2 + z^2 = 9$ for which $x^2 + y^2 \geq 2$, and let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F} = (-y, x, xyz).$$

Compute

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS,$$

where S is oriented so that the unit normal $\mathbf{N}$ to S points in the direction of $\mathbf{i}$ at the point $(3, 0, 0)$.

4. (a) Prove the identity

$$\nabla \times (f\nabla g) = \nabla f \times \nabla g,$$

   where f and g are twice differentiable functions of (x, y, z).

   (b) Let f and g be functions which are twice differentiable in a domain U of $\mathbf{R}^3$. Use (a) and Stokes' Theorem (or another method) to prove that

$$\oint_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = 0,$$

   where C is any closed curve which is the boundary of a parametrized surface S contained in U.

5. Compute, in two ways, the line integral

$$\oint_C (x^2 - y^2)dx - dy$$

where C is the boundary of the half disc $x^2 + y^2 \leq 1$ with $y \leq x$, oriented counter-clockwise.

   (a) By parametrizing the boundary curve (there are two pieces, a straight line segment and a semi-circle) and then evaluating the integral directly.

   (b) By applying Green's theorem, then evaluating the resulting double integral.

6. (a) For the surface S (helicoid or spiral ramp) swept out by the line segment joining the point $(2t, \cos t, \sin t)$ to $(2t, 0, 0)$ where $0 \leq t \leq \pi$, set up, but do not evaluate the definite integral which gives the area of this surface.

   (b) For the vector field $\mathbf{F} = (x, y, z)$ compute the flux of $\mathbf{F}$ through the surface of part (a). Assume the normal to the surface has a non-negative $\mathbf{k}$ component at $t = 0$.

7. Assume that a and b are fixed positive numbers. Find the extreme values of $\dfrac{x}{a} + \dfrac{y}{b}$ subject to the constraint $x^2 + y^2 = 1$.