MATH 264, Midterm version 1, solution outlines.

1. We decompose c as $c = c_1 + c_2 + c_3$, where c_1, c_2 and c_3 denote the line segments from $(0,0,0)$ to $(1,0,0)$, from $(1,0,0)$ to $(0,1,0)$ and from $(0,1,0)$ to $(0,0,0)$ respectively. We have the parametrizations

$$\mathbf{c_1}(t) = (t,0,0),\ \mathbf{c_2}(t) = (1-t,t,0),\ \mathbf{c_3}(t) = (0,1-t,0),\ 0 \le t \le 1,$$

with $||\mathbf{c_1'(t)}|| = ||\mathbf{c_3'(t)}|| = 1$ and $||\mathbf{c_2'(t)}|| = \sqrt{2}$. It follows that

$$\int_c (x+y)ds = \int_0^1 (t + \sqrt{2}(1-t+t) + 1 - t)dt = \sqrt{2} + 1.$$

2. We have

$$\mathbf{F} = \nabla V,\ V = x^2yz^3 + \sin(xz).$$

Therefore

$$\int_c \mathbf{F}d\mathbf{r} = V(\mathbf{c}(1)) - V(\mathbf{c}(0)) = V(2,2,0) - V(1,0,3) = \sin 3.$$

3. By symmetry,

$$\iint_S |xyz|dS = 4\iint_{S_1} xyzdS,$$

where S_1 denotes the portion of the paraboloid of revolution $z = x^2 + y^2$ which lies in the positive octant $x \ge 0,\ y \ge 0,\ z \ge 0$. We parametrize S_1 as the graph

$$\mathbf{X}(x,y) = (x,y,x^2+y^2),$$

where $(x,y) \in D$ and $D = \{(x,y)\,|\,x \ge 0,\ y \ge 0,\ x^2+y^2 \le 1\}$, so that

$$||\mathbf{X}_x \times \mathbf{X}_y|| = \sqrt{4x^2 + 4y^2 + 1}.$$

This gives

$$\iint_S |xyz|dS = 4\int_0^{\pi/2}\int_0^1 \cos\theta\sin\theta r^5\sqrt{1+4r^2}drd\theta$$

or

$$\iint_S |xyz|dS = 4\left(\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\right)\left(\int_0^1 r^5\sqrt{1+4r^2}dr\right).$$

Now,

$$\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = 1/2,$$

so that

$$\int_S |xyz| dS = 2\int_0^1 r^5\sqrt{1+4r^2}dr.$$

To evaluate the radial integral, we let $u = 1+4r^2$, which gives

$$\int_0^1 r^5\sqrt{1+4r^2}dr = \int_1^5 \frac{1}{16}(u-1)^2\frac{1}{8}\sqrt{u}\,du = \frac{1}{128}[\frac{2}{7}5^{\frac{7}{2}} - \frac{4}{5}5^{\frac{5}{2}} + \frac{2}{3}5^{\frac{3}{2}} + \frac{16}{105}],$$

and

$$\iint_S |xyz|dS = \frac{1}{64}[\frac{2}{7}5^{\frac{7}{2}} - \frac{4}{5}5^{\frac{5}{2}} + \frac{2}{3}5^{\frac{3}{2}} - \frac{16}{105}] = \frac{1}{64}\left(\sqrt{5}\,\frac{400}{21} - \frac{16}{105}\right).$$

.

4. We split our surface S as $S = S_1 + S_2 + S_3$, where S_1 and S_3 denote the lids $y^2+z^2 \leq 1, x = 1$ and $y^2+z^2 \leq 1, x = 0$ respectively, and S_2 denotes the cylinder $y^2+z^2 = 1, 0 \leq x \leq 1$. We have

$$\iint_{S_1} \mathbf{F}d\mathbf{S} = \iint_{z^2+y^2\leq 1} (z,y,1).(1,0,0)dzdy = \int_0^{2\pi}\int_0^1 r\cos\theta r dr d\theta = 0.$$

Likewise

$$\iint_{S_3} \mathbf{F}d\mathbf{S} = \iint_{z^2+y^2\leq 1} (0,y,0).(-1,0,0)dzdy = 0.$$

Finally, we parametrize S_2 by

$$\mathbf{X}(x,\theta) = (x, \cos\theta, \sin\theta),\ 0 \leq x \leq 1,\ 0 \leq \theta \leq 2\pi,$$

so that

$$\mathbf{X}_\theta \times \mathbf{X}_x = \cos\theta\mathbf{j} + \sin\theta\mathbf{k}.$$

We have

$$\iint_S \mathbf{F}d\mathbf{S} = \iint_{S_2} \mathbf{F}d\mathbf{S},$$

and

$$\iint_{S_2} \mathbf{F}d\mathbf{S} = \int_0^{2\pi}\int_0^1 (x\sin\theta, \cos\theta, x).(0,\cos\theta,\sin\theta)dxd\theta = \pi.$$