MATH 264, Midterm version 2, solution outlines.

1. We decompose c as $c = c_1 + c_2 + c_3$, where c_1, c_2 and c_3 denote the line segments from $(0,0,0)$ to $(1,0,0)$, from $(1,0,0)$ to $(0,0,1)$ and from $(0,0,1)$ to $(0,0,0)$ respectively. We have the parametrizations

$$\mathbf{c_1}(t) = (t,0,0),\ \mathbf{c_2}(t) = (1-t,0,t),\ \mathbf{c_3}(t) = (0,0,1-t),\ 0 \le t \le 1,$$

with $||\mathbf{c_1'(t)}|| = ||\mathbf{c_3'(t)}|| = 1$ and $||\mathbf{c_2'(t)}|| = \sqrt{2}$. It follows that

$$\int_c (x+2z)ds = \int_0^1 (t + \sqrt{2}(1-t+2t) + 2(1-t))dt = \frac{3}{2}(\sqrt{2}+1).$$

2. We have

$$\mathbf{F} = \nabla V,\ V = x^3y^2z + \exp(xy).$$

Therefore

$$\int_c \mathbf{F}d\mathbf{r} = V(\mathbf{c}(1)) - V(\mathbf{c}(0)) = V(3,-2,3) - V(2,4,0) = 324 + e^{-6} - e^8.$$

3. By symmetry,

$$\iint_S |xyz|dS = 4\iint_{S_1} xyzdS,$$

where S_1 denotes the portion of the paraboloid of revolution $y = x^2 + z^2$ which lies in the positive octant $x \ge 0,\ y \ge 0,\ z \ge 0$. We parametrize S_1 as the graph

$$\mathbf{X}(x,y) = (x, y, x^2 + y^2),$$

where $(x,y) \in D$ and $D = \{(x,y)\,|\, x \ge 0,\ y \ge 0,\ x^2 + z^2 \le 2\}$, so that

$$||\mathbf{X}_x \times \mathbf{X}_z|| = \sqrt{4x^2 + 4z^2 + 1}.$$

This gives

$$\iint_S |xyz|dS = 4\int_0^{\pi/2}\int_0^2 \cos\theta\sin\theta r^5\sqrt{1+4r^2}drd\theta$$

or

$$\iint_S |xyz|dS = 4\left(\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\right)\left(\int_0^2 r^5\sqrt{1+4r^2}dr\right).$$

Now,

$$\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta = 1/2,$$

so that

$$\int_S |xyz| dS = 2\int_0^2 r^5\sqrt{1+4r^2}dr.$$

To evaluate the radial integral, we let $u = 1 + 4r^2$, which gives

$$\int_0^1 r^5\sqrt{1+4r^2}dr = \int_1^{17} \frac{1}{16}(u-1)^2\frac{1}{8}\sqrt{u}\,du = \frac{1}{128}[\frac{2}{7}(17)^{\frac{7}{2}} - \frac{4}{5}(17)^{\frac{5}{2}} + \frac{2}{3}(17)^{\frac{3}{2}} - \frac{16}{105}],$$

and, after simplifications,

$$\iint_S |xyz| dS = \frac{1}{64}\left(\frac{124304}{105}\sqrt{17} - \frac{16}{105}\right).$$

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4. We split our surface S as $S = S_1 + S_2 + S_3$, where S_1 and S_3 denote the lids $x^2 + y^2 \leq 1, z = 1$ and $x^2 + y^2 \leq 1, z = 0$ respectively, and S_2 denotes the cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$. We have

$$\iint_{S_1} \mathbf{F} d\mathbf{S} = \iint_{x^2+y^2\leq 1} (x, y, 1).(0, 0, 1) dxdy = \pi.$$

Likewise

$$\iint_{S_3} \mathbf{F} d\mathbf{S} = \iint_{z^2+y^2\leq 1} (0, y, 0).(0, 0, -1) dzdy = 0.$$

Finally, we parametrize S_2 by

$$\mathbf{X}(z, \theta) = (\cos\theta, \sin\theta, z),\ 0 \leq z \leq 1,\ 0 \leq \theta \leq 2\pi,$$

so that

$$\mathbf{X}_\theta \times \mathbf{X}_z = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}.$$

We have

$$\iint_{S_2} \mathbf{F} d\mathbf{S} = \int_0^{2\pi}\int_0^1 (z\cos\theta, \sin\theta, z).(\cos\theta, \sin\theta, 0) dzd\theta = \frac{3\pi}{2},$$

and

$$\iint_S \mathbf{F} d\mathbf{S} = \pi + \frac{3\pi}{2} = \frac{5\pi}{2}.$$