Coordinate Transformations in Advanced Calculus

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This document summarizes the important coordinate transformations required in Advanced Calculus, and also states for reference the theorem required to use them to solve problems involving multiple integration. Memorizing the information in this document is strongly recommended; the best way to achieve this is to do many problems!

Plane-polar (2D)

- Coordinate variables: $r \in [0, +\infty]$, $\theta \in [0, 2\pi]$
- Mapping and inverse mapping from Cartesian: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x) \end{cases}$
- Jacobian determinant: r
- Regions generated by constant mapping variables:
  - r: circle of radius r
  - θ: half-line extending from the origin, having an angle of θ with respect to the $+x$ axis
- Use this transformation when the integrand and/or domain of integration involve:
  - Circles and other expressions similar to $x^2 + y^2$
  - Regions between two rotated line segments

Cylindrical (3D)

- Coordinate variables: $r \in [0, +\infty]$, $\theta \in [0, 2\pi]$, $z \in [-\infty, +\infty]$
- Mapping and inverse mapping from Cartesian: same as Plane Polar (2D), with the addition of $z = z$
- Jacobian determinant: r
- Regions generated by constant mapping variables:
  - r: cylinder of base radius r
  - θ: vertical half-plane
  - z: horizontal plane
- Use this transformation when the integrand and/or domain of integration involve:
  - Cylinders, cones, and other expressions similar to $x^2 + y^2$
  - Regions between two rotated vertical planes
- Note (for electrical engineering students): in electromagnetic fields literature and classes (e.g. ECSE 351), r is denoted ρ; also, θ is denoted ϕ

Spherical (3D)

- Coordinate variables: $\rho \in [0, +\infty]$, $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$
- Mapping and inverse mapping from Cartesian:
$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \phi = \arctan((\sqrt{x^2 + y^2})/z) \\ \theta = \arctan(y/x) \end{cases}$$
- Jacobian determinant: $\rho^2 \sin\phi$
- Regions generated by constant mapping variables:
  - ρ: sphere of radius ρ
  - ϕ: cone with an angle of ϕ from the $+z$ axis
  - θ: vertical half-plane
- Use this transformation when the integrand and/or domain of integration involve:
  - Spheres and other expressions similar to $x^2 + y^2 + z^2$
  - Cone-shaped sections of spheres, such as "ice cream cone"-shaped regions (i.e. cones with a spherical "cap")
  - Spherical regions between two half-planes
- Note (for electrical engineering students): in electromagnetic fields literature and classes (e.g. ECSE 351), ρ is denoted r; also, ϕ and θ are switched (ϕ is denoted θ, and vice-versa)

Elliptical / Ellipsoidal (2D and 3D)

- Coordinate variables: $u, v, w \in [-\infty, +\infty]$ (in the 2D case, use only u and v)
- Mapping and inverse mapping: $\begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \leftrightarrow \begin{cases} u = x/a \\ v = y/b \\ w = z/c \end{cases}$
  where a, b, and c are non-zero constants
- Jacobian determinant: ab (2D case), abc (3D case)
- Regions generated by constant mapping variables: horizontal or vertical planes
- Use this transformation when the integrand and/or domain of integration involve ellipses (2D, $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$) or ellipsoids (3D, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$)

Bounded Cartesian Functions (2D and 3D)

- Coordinate variables: u, v, w (in the 2D case, use only u and v)
- Mapping and inverse mapping: $\begin{cases} u = f(x, y, z) \\ v = g(x, y, z) \\ w = h(x, y, z) \end{cases}$
  where f, g, and h are functions (the inverse mapping depends on what these functions are, and requires solving the three equations for x, y, and z in terms of u, v, and w)
- Jacobian determinant: Compute using the definition of the Jacobian given in the theorem on page 5 of this document
- Regions generated by constant mapping variables: level surfaces or curves of the functions f, g, or h (e.g. for $u = c$ where c is a constant, $f(x, y, z) = c$)
- Use this transformation when the domain of integration is defined in the form $\begin{cases} c_{f1} \le f(x, y, z) \le c_{f2} \\ c_{g1} \le g(x, y, z) \le c_{g2} \\ c_{h1} \le h(x, y, z) \le c_{h2} \end{cases}$ where the c_k are constants

The Change of Variables Theorem

Let:

- $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \leftrightarrow \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ denote a one-to one (i.e. invertible) 2D coordinate transformation.
- $\frac{\partial(x,y)}{\partial(u,v)}$ be a `2x2` matrix defined as follows:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

- D denote a region (e.g. a surface) parametrized using the x and y coordinate system.
- D' denote the region D, parametrized using the u and v coordinate system.

Then:

- Matrix $\frac{\partial(x,y)}{\partial(u,v)}$ is called the Jacobian of the coordinate transformation.
- The determinant of this matrix, denoted $\det[\frac{\partial(x,y)}{\partial(u,v)}]$, is called the Jacobian determinant, and it is guaranteed to be nonzero (if it is zero, the coordinate transformation is not one-to-one and the conclusions of this theorem may not hold true).
- The following change of variables formula holds:

$$\int\int_D f(x,y)\,dx\,dy = \int\int_{D'} f(x(u,v),y(u,v))\det[\frac{\partial(x,y)}{\partial(u,v)}]\,du\,dv$$

- A similar change of variables formula holds for 3D and higher-dimensional coordinate transformations (add more variables in the appropriate places).