Fundamental Theorems of Vector Calculus

by Sacha Nandlall
T.A. for MATH 264 (Advanced Calculus), McGill University

Email: sacha.nandlall@mail.mcgill.ca
Website: http://www.resanova.com/teaching/calculus/

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This document summarizes the different versions of the fundamental theorem of calculus, particularly those relevant to vector calculus in 1D, 2D, and 3D. It also defines and illustrates the rule used in some of these theorems for orienting boundaries of 2D surfaces.

Orienting boundaries of 3D surfaces using the right-hand rule

- Consider an open (i.e. not closed) surface S with unit normal $\widehat{N}$ and differential surface element $d\vec{S} = \widehat{N} dS$.
- Let ∂D denote the curve bounding this surface, and distinguish two possible orientations of this boundary curve: positive, denoted ∂D^+; and negative, denoted ∂D^-.
- The directions of ∂D^+ and ∂D^- may be found via the following right-hand rule:
  1. Point the right hand's thumb in the direction of $d\vec{S}$ (or of $\widehat{N}$).
  2. Place the right hand's palm against the boundary's surface.
  3. Then the other fingers of the hand point in the direction of the positively-oriented boundary, ∂D^+; the negatively-oriented ∂D^- boundary points in the opposite direction.

See the Figure on the next page for illustrations of the right-hand rule.

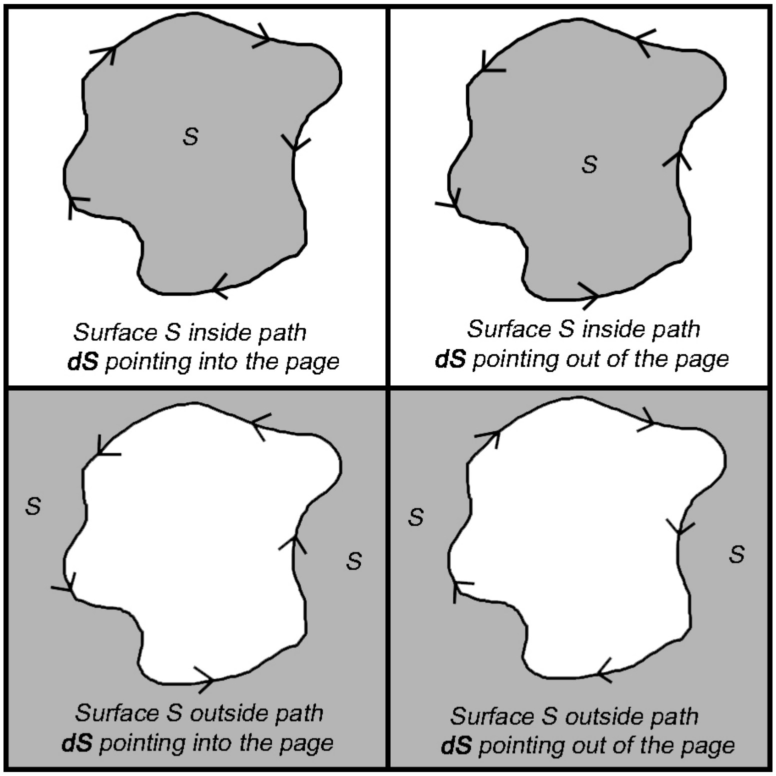

Figure 1: Applying the right-hand rule to the orientation of 3D surfaces

Fundamental Theorem of Integral Calculus (1D)

Let f be a scalar differentiable function. Then:

$$\int_a^b \left(\frac{df(t)}{dt}\right) dt = f(b) - f(a)$$

Fundamental Theorem for Conservative Fields (1D)

Let ϕ be a scalar field (i.e. a scalar function defined over 3D space) and $C \sim \vec{r}(t)$, t from a to b be a vector curve. Then:

$$\int_C (\text{grad } \phi) \bullet d\vec{r} = \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

Stokes' and Green's Theorems (2D)

Let $\vec{F}$ be a vector function, S be an open (i.e. not closed) surface with differential surface element $d\vec{S} = \widehat{N} dS$, and ∂D^+ be the positively-oriented boundary of S. Then by Stokes' theorem:

$$\iint_S (\text{curl } \vec{F}) \bullet d\vec{S} = \oint_{\partial D^+} \vec{F} \bullet d\vec{r}$$

Green's theorem is a special case of Stokes' theorem that applies when $\vec{F}$ lies in the xy plane, i.e. $\vec{F} = F_1 \widehat{\imath} + F_2 \widehat{\jmath}$. In Stokes' theorem, use $\widehat{N} = \pm \widehat{k}$ (i.e. $d\vec{S} = \pm \widehat{k}\, dS$) and

$$\text{curl } \vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \widehat{k}$$

Divergence Theorem (2D)

Let $\vec{F}$ be a vector function, S be an open (i.e. not closed) surface with differential surface element $d\vec{S} = \widehat{N} dS$, and ∂D^+ be the positively-oriented boundary of S. Then:

$$\iint_S (\text{div } \vec{F})\, dS = \oint_{\partial D^+} (\vec{F} \bullet \widehat{N}) ds$$

Divergence Theorem (3D)

Let $\vec{F}$ be a vector function, ϕ be a scalar field, and V be a volume bounded by surface S. Then:

$$\iiint_V (\text{div } \vec{F})\, dV = \oiint_S \vec{F} \bullet d\vec{S}$$

$$\iiint_V (\text{curl } \vec{F})\, dV = -\oiint_S \vec{F} \times d\vec{S}$$

$$\iiint_V (\text{grad } \phi)\, dV = \oiint_S \phi\, d\vec{S}$$