TUTORIAL #1
Multiple Integration

Presented by Sacha Nandlall
T.A. for MATH 264 (Advanced Calculus)

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at 11:35 A.M. in Wilson 103

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WELCOME!

- Email: sacha.nandlall@mail.mcgill.ca
- Website address: http://www.ece.mcgill.ca/~snandl/math264/
  - Slides
  - Past problems and solutions (old finals, midterms, and assignments)
  - Maple code
  - Useful reference material

IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

MULTIPLE INTEGRATION

- Representing the multiple integral
  - Double (surface, 2D) integral: $\int\int_S f(u,v)dS$
    - Sometimes *dA* ("area") is used instead of *dS*
  - Triple (volume, 3D) integral: $\int\int\int_V f(u,v,w)dV$
  - And so on for higher dimensions…
- Terminology
  - *S* and *V* are the regions/domains of integration
  - *f(u,v)* and *f(u,v,w)* are the integrands
  - *dS* and *dV* are differential elements of surface/volume

MULTIPLE INTEGRATION

- To evaluate multiple integrals, use iteration
  - Review Intermediate Calculus
  - MATH 260/262 or equivalent
- Sometimes, the region of integration is hard to describe using Cartesian coordinates
  - Spheres, cylinders, ellipses, ellipsoids, etc.
  - In this case, we can apply a coordinate transform to the integral

COORDINATE TRANSFORMS

- Applying a coordinate transform involves:
  - Reparametrizing the domain of integration using the new coordinate system
  - Changing the variables in the integrand to those of the new coordinate system
  - Rewriting the differential element (*dS*, *dV,* etc.)
- This can make the integral easier to do
  - We will now give a formula for applying a coordinate transform

THE JACOBIAN MATRIX

- Consider the mapping $(x_k)_{k=1}^{m} \to (u_k)_{k=1}^{n}$
  - In long form: $\begin{cases} x_1 = x_1(u_1, u_2, \ldots, u_n) \\ x_2 = x_2(u_1, u_2, \ldots, u_n) \\ \cdots \\ x_m = x_m(u_1, u_2, \ldots, u_n) \end{cases}$
- The associated Jacobian matrix is:

$$\frac{\partial(x_1, x_2, \ldots, x_m)}{\partial(u_1, u_2, \ldots, u_n)} = \left(\frac{\partial x_i}{\partial u_j}\right)_{(1 \le i \le m) \wedge (1 \le j \le n)} = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial u_1} & \frac{\partial x_m}{\partial u_2} & \cdots & \frac{\partial x_m}{\partial u_n} \end{pmatrix}$$

THE CHANGE OF VARIABLES FORMULA

- Consider the 2D mapping $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$
- Then for a 2D (double) integral expressed as a function of x and y:

$$\int\int_D f(x,y)\,dx\,dy = \int\int_{D'} f(x(u,v),y(u,v))\,\det[\frac{\partial(x,y)}{\partial(u,v)}]\,du\,dv$$

- A similar formula holds for 3D and higher-dimensional integrals (add more variables)

METHOD TO EVALUATE MULTIPLE INTEGRALS

1. Whenever possible, sketch the region of integration D
2. Choose a coordinate system, depending on the form of D and/or the form of the integrand
   – Refer to the handout on the webpage
3. Describe (i.e. parametrize) D using the chosen coordinate system
4. Express the integrand (the function being integrated) in terms of this coordinate system

METHOD TO EVALUATE MULTIPLE INTEGRALS

5. Write the expression for the differential element (*dS*, *dV*, etc.)
   - Refer to the formula given previously (multiply by the determinant of the transformation's Jacobian matrix)
   - The Jacobian determinant can be quoted from memory or else calculated
6. Iterate the integral and evaluate it