TUTORIAL #3
3D Surface Integrals

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T.A. for MATH 264 (Advanced Calculus)
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IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

INTEGRATING OVER SURFACES IN 3D

- Let's revisit surface integrals: $\iint_S f(u,v)\,dS$
- We've already seen how to integrate over 2D surfaces in the *xy* plane
- What if the surface of integration is 3D?
  - In particular, the area of a surface *S* is $\iint_S dS$
  - We need other techniques to do this integral if *S* is a 3D surface (like a paraboloid, sphere, etc.)
  - Let's first examine how to describe 3D surfaces

PARAMETRIZING 3D SURFACES

- 3D surfaces can be parametrized as follows:

$$\vec{r}_s(u,v) = [x(u,v)]\hat{\imath} + [y(u,v)]\hat{\jmath} + [z(u,v)]\hat{k} \text{ where } (u,v) \in D$$

  - *u* and *v* are two parameters
    - IMPORTANT: any surface, including 3D ones, only need two parameters to describe them, not three!
  - $\boldsymbol{r_s}$ is a vector function of the two parameters
    - Gives the Cartesian (*x*, *y*, *z*) coordinates of points on the surface as a vector, as a function of *u* and *v*
  - *D* provides bounds on parameters *u* and *v*

3D DIFFERENTIAL SURFACE ELEMENTS

- Because our surface *S* is in 3D, we need a new formula for differential element *dS*
  - For reasons we'll see later, we'll first define a vector differential surface element ***dS***:

$$d\vec{S} = \pm \left(\frac{\partial \vec{r_s}(u,v)}{\partial u} \times \frac{\partial \vec{r_s}(u,v)}{\partial v} \right) du\, dv$$

  - The (scalar) differential surface element *dS* is then the magnitude of the vector element:

$$dS = |d\vec{S}| = \left| \frac{\partial \vec{r_s}(u,v)}{\partial u} \times \frac{\partial \vec{r_s}(u,v)}{\partial v} \right| du\, dv$$

METHOD TO EVALUATE 3D SURFACE INTEGRALS

- Just follow the same 6-step method as before (see Tutorial 1), with these changes:
  - Step 2: pick a 3D coordinate system, but eliminate one of the coordinate variables by expressing it as a function of the other two
    - The surface's equation will allow you to do this
  - Step 3: parametrize surface S by describing it as a vector function $\boldsymbol{r_s}(u,v)$ as shown before
  - Step 5: use the formula for dS listed previously