TUTORIAL #4
Flux Integrals and Vector Fields

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T.A. for MATH 264 (Advanced Calculus)

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IMPORTANT DISCLAIMER

- Great effort has been made to ensure the material presented here is factually correct
- However, students (that's you!) are fully responsible for verifying all material (theorems, formulas, solutions, etc.) presented on their own

VECTOR FUNCTIONS AND FIELDS

- A vector function is a function whose inputs and outputs are vectors
- A vector field is a vector function that maps from R^n to R^n (same spatial dimensions)
  - Common: R^2 to R^2 (2D) and R^3 to R^3 (3D)
  - Notation
    - 3D: $\vec{F}(x, y, z) = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$
    - 2D: same as 3D with x and y only and $F_3 = 0$
    - n-dimensional: $\mathbf{F}(x_1, x_2, \ldots, x_n)$ or $\vec{F}(x_1, x_2, \ldots, x_n)$

DIFFERENTIAL OPERATORS

- What is an operator?
  - An entity that inputs and outputs functions
  - Differential operators involve 2D or 3D vector functions, and are typically defined in Cartesian
  - Functions have to be converted to Cartesian first before applying differential operators
- The basic differential operator is the del or nabla operator (denoted ∇)
  - It is defined in Cartesian as $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

VECTOR DIFFERENTIAL OPERATORS

- These operators act on vector fields
- Divergence: $\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
  - Outputs a scalar function
- Curl:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

  - Outputs another vector field
  - For 2D vector fields (*x* and *y* only, $F_3 = 0$):

$$\text{curl}\,\vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \widehat{k}$$

SCALAR DIFFERENTIAL OPERATORS

- These operators act on scalar functions $\varphi(x, y, z)$
- Gradient: $\text{grad}\ \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$
  - Outputs a vector field
- Laplacian: $$\nabla^2\phi = \nabla\cdot\nabla\phi = \text{div grad}\ \phi$$ $$= \left(\frac{\partial\phi}{\partial x}\right)^2\hat{\imath} + \left(\frac{\partial\phi}{\partial y}\right)^2\hat{\jmath} + \left(\frac{\partial\phi}{\partial z}\right)^2\hat{k}$$
  - Outputs a scalar function

FLUX INTEGRALS

- Let's extend the 3D surface integrals we did last time and replace the integrand by a vector function ***F***
  - We'd like to measure "how much" of the field ***F*** passes through our surface
- Consider $\int\int_S \vec{F} \bullet d\vec{S}$
  - Differential surface element ***dS*** is now a vector
  - We call this vector 3D surface integral the flux of ***F*** through the surface *S*

VECTOR DIFFERENTIAL SURFACE ELEMENTS

- We want only the normal (perpendicular) component of ***F*** through surface *S*
- It turns out that:

$$d\vec{S} = \pm \left(\frac{\partial \vec{r_s}(u,v)}{\partial u} \times \frac{\partial \vec{r_s}(u,v)}{\partial v} \right) du\, dv$$

$$d\vec{S} = \hat{N}\, dS$$

  – *dS* is the scalar differential element for 3D surfaces (magnitude of the ***dS*** vector)
  – ***N*** is a unit normal to the surface *S*

SOME REMARKS...

- ***N*** and ***dS*** can be oriented in two directions
  - This is why there is a plus/minus sign!
  - For flux integrals, if not told otherwise, assume a flux directed out of the surface
- ***N*** is a unit normal
  - Always normalize it to a magnitude of 1
- ***N*** can sometimes be found by inspection
  - Examples: planes, spheres, cylinders

GAUSS'S LAW AND FLUX INTEGRALS

- Consider a closed surface *S* whose boundary doesn't contain the origin
- Let ***F*** be the following vector function:

$$\vec{F}(x,y,z) = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\hat{r}}{|\vec{r}|^2}$$

- Then the outward-directed flux of ***F*** through surface *S* is:

$$\int\int_S \vec{F} \bullet d\vec{S} = \begin{cases} 4\pi & \text{if } S \text{ encloses the origin} \\ 0 & \text{otherwise} \end{cases}$$

- This is Gauss's law; memorize the result and quote it if you need to