TUTORIAL 1 SOLUTIONS (SEPTEMBER 14, 2006) – VERSION 1

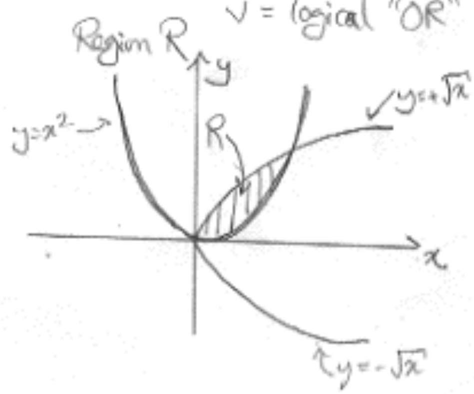
Problem 9, Section 14.2 (Adams)

Question: Evaluate $\iint_R xy^2 \, dA$ where R is the finite region in the first quadrant bounded by $y=x^2$ and $x=y^2$

Solution:

Follow the 6-step method described in the Tutorial 1 slides.

Note
$\wedge$ = logical "AND"
$\vee$ = logical "OR"



1) Sketch the region of integration R

$\rightarrow y = x^2$ is a parabola

$\rightarrow x = y^2 \Rightarrow (y=-\sqrt{x}) \vee (y=+\sqrt{x})$ ← denotes "or"

$\Rightarrow$ horizontally-oriented parabola

2) Choose a coordinate system

Cartesian is fine here

3) Parametrize R in this coordinate system

This means, find a set (or sets) of inequalities that cover all of R.

From our sketch, the most obvious way to do this is to find the points where the $y=x^2$ and $y=\sqrt{x}$ curves intersect:

$$(y=x^2) \wedge (y=\sqrt{x}) \Rightarrow x^2=\sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = 0$$
denotes "and"
$$\Rightarrow x(x^3-1)=0 \Rightarrow (x=0) \vee (x^3=1)$$
$$\Rightarrow (x=0) \vee (x=\sqrt[3]{1}=1) \Rightarrow x \in \{0,1\}$$

Thus $0 \le x \le 1$.

From our sketch, we can see that for $0 \le x \le 1$, we have $x^2 \le y \le \sqrt{x}$ (not $\sqrt{x} \le y \le x^2$, careful)

$\therefore$ our parametrization of the region is $R: \begin{cases} 0 \le x \le 1 \\ x^2 \le y \le \sqrt{x} \end{cases}$

denotes "therefore"

$(\rightarrow)$

Some remarks about this process:

- We could just as well have used the parametrization
  $$R: \begin{cases} y^2 \le x \le \sqrt{y} \\ 0 \le y \le 1 \end{cases}$$
  Either way is correct and will give the same results.
  In this problem, both will be equally easy to use, but in other questions, one way might be harder than another
- Whatever description you do come up with, one of the inequalities has to be of the form $a \le (\text{variable}) \le b$ where a and b are constants. Otherwise, you won't be able to do the integral.

4) Express the integrand in terms of the chosen coordinate system

The integrand is already in Cartesian: xy^2

5) Write the expression for the differential element

$dA = dx\,dy$ (no factor to multiply by, as we haven't changed coordinate systems)

6) Iterate the integral and evaluate it

This combines steps 3 to 5

$$\begin{aligned}
\iint_R xy^2\,dA &= \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy\; xy^2 \\
&= \int_0^1 x \left[\int_{x^2}^{\sqrt{x}} y^2\right] dx = \int_0^1 x \left[y^3/3\right]_{x^2}^{\sqrt{x}} dx \\
&= (1/3)\int_0^1 x\left(x^{3/2} - x^6\right) dx \\
&= (1/3)\int_0^1 \left(x^{5/2} - x^7\right) dx = (1/3)\left[\frac{x^{7/2}}{7/2} - \frac{x^8}{8}\right]_0^1 \\
&= (1/3)\left[\left[(2/7)\cdot(1)^{7/2} - 1/8\right] - [0-0]\right] \\
&= (1/3)\left[16/56 - 7/56\right] = (1/3)(9/56) \\
&= 3/56
\end{aligned}$$

NOTE: the outermost (last) integral must be the one with the two constant bounds

$$\therefore \iint_R xy^2\,dA = \boxed{3/56}$$

Remember you can always check this with Maple!
See the Maple code file for Tutorial 1 on the webpage.

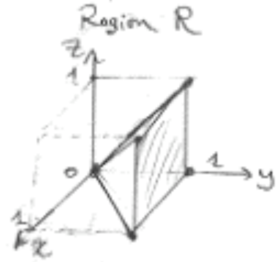
TUTORIAL 1 SOLUTIONS (SEPTEMBER 14, 2006) – VERSION 1

Problem 9, Section 14.5 (Adams)

Question: Evaluate $\iiint_R \sin(\pi y^3)\,dV$ over the pyramid w/ vertices at $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(1,1,1)$, $(0,1,1)$

Solution:

1) Sketch the region
R is sketched at the left; simply plot the given vertices and join them up



You can check the sketch with Maple too

2) Choose a coordinate system
Cartesian will do just fine

3) Parametrize R
This is a bit trickier to visualize, but it can be thought through carefully.
The planes bounding the pyramid are:
$z=0$, $y=x$, $x=0$, $z=y$, and $y=1$
(Check that each of these planes corresponds to a face of the pyramid in the sketch above to convince yourself)
Making use of these equations, we can describe the pyramid as follows:

$$R: \begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \\ 0 \le z \le y \end{cases}$$

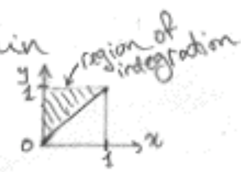
(other descriptions exist, but all will give the same answer; as we will see below however, some descriptions might result in easier integrals than others...)

4) Integrand: $\sin(\pi y^3)$

5) $dV = dx\,dy\,dz$

6)
$$\iiint_R \sin(\pi y^3)\,dV = \int_0^1 dx \int_x^1 dy \int_0^y dz\ \sin(\pi y^3)$$
$$= \int_0^1 dx \int_x^1 \sin(\pi y^3)\,dy \int_0^y dz$$
$$= \int_0^1 dx \int_x^1 y \sin(\pi y^3)\,dy$$

This integral is hard! However, let's try changing the domain of integration as follows:
$$\begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \end{cases} \equiv \begin{cases} 0 \le y \le 1 \\ 0 \le x \le y \end{cases}$$



Ergo $\iiint_R \sin(\pi y^3)\,dV = \int_0^1 dy \int_0^y dx\ y\sin(\pi y^3) = \int_0^1 y^2 \sin(\pi y^3)\,dy$

Now $\int y^2 \sin(\pi y^3)\,dy = (1/3\pi)\int \sin(u)\,du$
$= -(1/3\pi)\cos(u) = -(1/3\pi)\cos(\pi y^3)$

Let $u = \pi y^3$
Then $du = 3\pi y^2\,dy \Rightarrow y^2\,dy = \frac{du}{3\pi}$

Thus $\int_0^1 y^2 \sin(\pi y^3)\,dy = \left[-(1/3\pi)\cos(\pi y^3)\right]_0^1 = -(1/3\pi)\cos(\pi) + (1/3\pi)\cos 0$
$= 1/3\pi + 1/3\pi = 2/3\pi$

$\therefore \iiint_R \sin(\pi y^3)\,dV = \boxed{2/3\pi}$

Check it with Maple!

Note: we could have used $\begin{cases} 0 \le x \le y \\ 0 \le y \le 1 \\ 0 \le z \le y \end{cases}$ to describe R right away; this illustrates that sometimes, the order of integration does matter

TUTORIAL 1 SOLUTIONS (SEPTEMBER 14, 2006) – VERSION 1

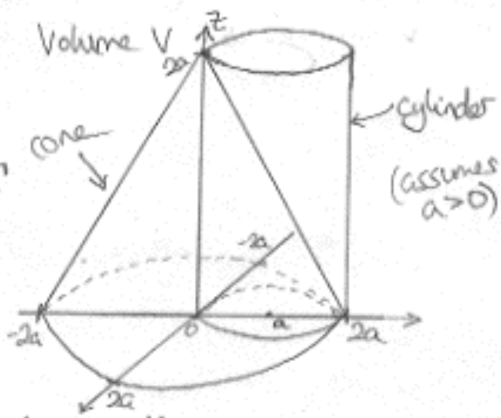
Problem 19, Section 14.6 (Adams)

Question: Find the volume of the region V above the xy-plane, inside the cone $z = 2a - \sqrt{x^2+y^2}$, and inside the cylinder $x^2+y^2 = 2ay$

Solution:

Recall that the volume of V is $\iiint_V dV$ (i.e. integrate 1 over the region). Follow the same steps as the last problem

1) Sketch the region



There are three parts to this:

→ Cone: $z = 2a - \sqrt{x^2+y^2} \Rightarrow (z-2a) = -\sqrt{x^2+y^2}$

We know that $z = \sqrt{x^2+y^2}$ is a 45° cone oriented upwards.

$z = -\sqrt{x^2+y^2}$ is the same cone, but oriented downwards

Finally, $(z-2a) = -\sqrt{x^2+y^2}$ is a downward-pointing cone shifted upward by $2a$ units (or downward if $a<0$)

The cone intersects the xy-plane for $(z=0) \wedge (z = 2a - \sqrt{x^2+y^2})$

$\Rightarrow 0 = 2a - \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = (2a)^2$ (circle of radius $|2a|$ centered at origin)

→ Cylinder: A right cylinder's equation is that of a circle with no restriction in z; i.e. $(x-h)^2+(y-k)^2 = r^2$ denotes a cylinder of radius r whose central axis passes through the point $(h,k,0)$

In this case, $x^2+y^2 = 2ay \Leftrightarrow x^2+y^2-2ay = 0$

$\Leftrightarrow x^2+(y-a)^2 - a^2 = 0$ (complete the square)

$\Leftrightarrow x^2+(y-a)^2 = a^2$

This is a cylinder of radius a with a central axis through $(0,a,0)$

→ Above the xy plane: this means that $z>0$.

Note that if $a \leq 0$, the cylinder is below $z>0$, and hence the volume V will be empty

Combining all this, we get the sketch above for $a>0$

(→)

TUTORIAL 1 SOLUTIONS (SEPTEMBER 14, 2006) – VERSION 1

2) Choose a coordinate system
Cylindrical is a good choice here, because of the presence of the cylinder and "x^2+y^2" terms in the equations describing the region

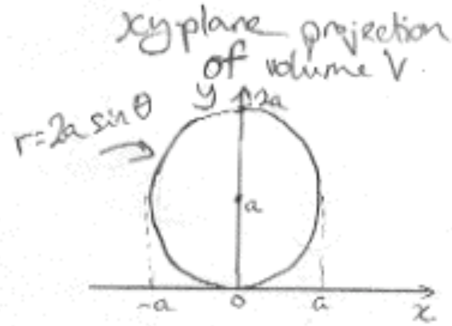
3) Parametrize V
If $a \le 0$, $V = \emptyset$ (the null set, i.e. V is empty)
Now assume $a > 0$. We want bounds on the parameters r, θ, and z. To do this, we'll make the following observations using our sketch:
→ z seems to lie between the cone's base at $z=0$ and the lateral face of the cone
→ r and θ only depend on x and y, so to ensure that they span the full range of the volume, we could essentially project the volume onto the xy plane. This would be like flattening V onto the xy plane and parametrizing the 2D shape that results using r and θ (plane polar)
This trick works in general when using cylindrical coordinates, and we'll illustrate it shortly

Let's begin by parametrizing the equations of the cone and cylinder in cylindrical ($x = r\cos\theta$, $y = r\sin\theta$, $z=z$)
→ Cone: $z = 2a - \sqrt{x^2+y^2} = 2a - \sqrt{r^2} = 2a - r$ (since $r \ge 0$)
→ Cylinder: $x^2+y^2 = 2ay \Rightarrow r^2 = 2a\, r\sin\theta \Leftrightarrow r = 2a\sin\theta$
→ xy plane: $z = 0$

We observed z lies between the xy plane and the cone, and so $0 \le z \le 2a - r$ ($z=0$ is the xy plane, $z = 2a-r$ is the cone)

Now what happens when we project the volume V onto the xy plane? From our sketch, we see that when the smoke clears, the projection will basically be the base of the cylinder. This takes some visualization to see, but be sure you can convince yourself of this. (→)

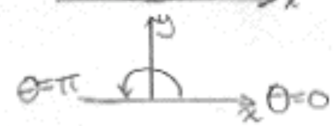
The projection is the same shape as the cylinders base, i.e. a circle of radius a centered at $(0, a)$



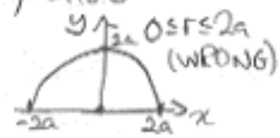
The equation for this circle in cylindrical is $r = 2a \sin\theta$ (same as the cylinder, as putting $z=0$ in the cylinder's equation changes nothing)

This gives us our bounds on r: $0 \le r \le 2a\sin\theta$ (r extends out from the origin to the circle, as shown at the right)

Also, we can see that to sweep the circle, we need $0 \le \theta \le \pi$, since it lies in the top half of the xy plane



→ Note: parametrizations such as $a \le r \le 2a$ or $0 \le r \le 2a$ may look tempting, but they won't work. Remember, r starts from the origin, not the circle's center; also $0 \le r \le 2a$ would sweep an entire half circle as shown at the right



Also, $0 \le \theta \le 2\pi$ would sweep the entire plane, rather than only the top half as required.

Whew! To summarize, our parametrization of V is:

$$V: \begin{cases} 0 \le r \le 2a\sin\theta \\ 0 \le \theta \le \pi \\ 0 \le z \le 2a - r \end{cases}$$

4) Express the integrand in cylindrical coordinates:
The integrand is just 1.

5) Write the expression for the differential element
The Jacobian determinant for Cartesian to cylindrical is r (should memorize this)

$\therefore dV = r\,dr\,d\theta\,dz$ (→)

6) Evaluate the integral

$$\iiint_D (1)\,dV = \int_0^\pi d\theta \int_0^{2a\sin\theta} dr \int_0^{2a-r} dz \cdot r$$

the Jacobian determinant, don't forget!

$$= \int_0^\pi d\theta \int_0^{2a\sin\theta} r\,[z]_0^{2a-r}\,dr$$

Note: the z integral has bounds that depend on r, so it must go inside the r integral

$$= \int_0^\pi d\theta \int_0^{2a\sin\theta} r[(2a-r)-(0)]\,dr$$

$$= \int_0^\pi d\theta \int_0^{2a\sin\theta} (2ar - r^2)\,dr$$

The r integral goes inside the θ integral for the same reason

$$= \int_0^\pi \left[ar^2 - r^3/3\right]_0^{2a\sin\theta} d\theta$$

$$= \int_0^\pi \left[\left[a(2a\sin\theta)^2 - (2a\sin\theta)^3/3\right] - [0-0]\right] d\theta$$

$$= \int_0^\pi \left[4a^3\sin^2\theta - (8a^3/3)\sin^3\theta\right] d\theta$$

$$= 4a^3 \int_0^\pi \sin^2\theta\,d\theta - (8a^3/3)\int_0^\pi \sin^3\theta\,d\theta$$

refer to your first calculus classes if you're not sure how to do these integrals

$$= 4a^3\left[\theta/2 - (1/2)\sin(2\theta)\right]_0^\pi - (8a^3/3)\left[-(1/3)\sin^2\theta\cos\theta - (2/3)\cos\theta\right]_0^\pi$$

$$= 4a^3\left[\left[\pi/2 - (1/2)\sin(2\pi)\right] - \left[0 - (1/2)\sin(0)\right]\right] - (8a^3/3)\left[\left[-(1/3)\sin^2(\pi)\cos(\pi) - (2/3)\cos(\pi)\right] - \left[-(1/3)\sin^2(0)\cos(0) - (2/3)\cos(0)\right]\right]$$

$$= 4a^3[\pi/2] - (8a^3/3)\left[(-2/3)(1) - (-2/3)(1)\right]$$

$$= (2\pi - 32/9)a^3$$

$\therefore$ the volume of the region is $(2\pi - 32/9)a^3$ units^3