MATH 264 (Advanced Calculus), Fall 2006, McGill University
Thursday Tutorial, 11:35 A.M. in Wilson 103
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TUTORIAL 10 SOLUTIONS (NOVEMBER 9, 2006) – VERSION 2

Problem 4 (Part A) of section 17.3 (Greenberg)

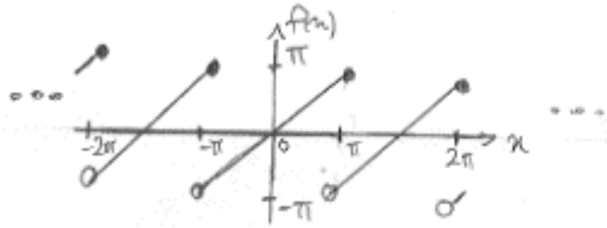
Question: Find the Fourier series of the following periodic function, specified over one period:

4a) $f(x) = x \quad , x \in (-\pi, \pi]$

Solution:

4a) $T = \pi - (-\pi) = 2\pi$

$\omega_0 = 2\pi/T = 1$



$$c_n = (1/2\pi) \int_{-\pi}^{\pi} x e^{-in \cdot 1 \cdot x} dx$$

Let $I(n,x) = \int x e^{-inx} dx \Rightarrow c_n = (1/2\pi)[I(n,\pi) - I(n,-\pi)]$

$I(0,x) = \int x \, dx = x^2/2$

$$I(n,x)\Big|_{n\neq 0} = x \cdot (-1/in) e^{-inx} - 1 \cdot (-1/in)^2 e^{-inx}$$
$$= [i(x/n) + (1/n^2)] e^{-inx}$$

u	dv
x	e^{-inx}
1	$(-1/in)e^{-inx}$
0	$(-1/in)^2 e^{-inx}$

So $c_0 = (1/2\pi)[\pi^2/2 - (-\pi)^2/2] = 0$ (yes; average value of $f(x)$ is zero)

and $c_n\Big|_{n\neq 0} = (1/2\pi)\left[(i(\pi/n) + (1/n^2)) e^{-in\pi} - (i(-\pi/n) + (1/n^2)) e^{-in(-\pi)} \right]$

Since $e^{in\pi} = (e^{i\pi})^n = (-1)^n$ and $e^{-in\pi} = (e^{-i\pi})^n = (-1)^n$

$$c_n\Big|_{n\neq 0} = (1/2\pi) \cdot (-1)^n \cdot [i\pi/n + 1/n^2 + i\pi/n - 1/n^2]$$
$$= i((-1)^n/n)$$

$\therefore c_n = \begin{cases} 0 & \text{if } n=0 \\ i[(-1)^n/n] & \text{otherwise} \end{cases} \Rightarrow \quad a_n = Re(c_n) = 0$

$b_n = Im(c_n) = \begin{cases} 0 & \text{if } n=0 \\ (-1)^n/n & \text{otherwise} \end{cases}$

(check: $c_n = (c_{-n})^*$)

$$\therefore f(x) = 0 + \sum_{n=1}^{+\infty} \left[2 \cdot 0 \cdot \cos(n\omega_0 x) - 2 \cdot [(-1)^n/n] \cdot \sin(n\omega_0 x) \right]$$
$$= \boxed{\sum_{n=1}^{+\infty} \left(\frac{2 \cdot (-1)^{n+1}}{n} \right) \sin(nx)}$$

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Problem 36, MATH 264 Assignment #1 (Fall 2006)

Question: Expand the function $f(x) = \cos x$, $0 < x < \pi$, in a Fourier sine series

Solution:

The odd extension of $f(x)$, denoted $g(x)$ here, is

$$g(x) = \begin{cases} -f(-x) & \text{if } -\pi < x < 0 \\ f(x) & \text{if } 0 < x < \pi \end{cases} = \begin{cases} -\cos(x) & \text{if } -\pi < x < 0 \\ \cos(x) & \text{if } 0 < x < \pi \end{cases}$$

(note that $\cos(-x) = \cos(x)$)

Now expand $g(x)$ into a Fourier series:

$T = \pi - (-\pi) = 2\pi$, $\omega_0 = 2\pi/T = 1$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) e^{-in(1)x}\,dx$$

$$= \frac{1}{2\pi}\left[-\int_{-\pi}^{0} \cos(x) e^{-inx}\,dx + \int_{0}^{\pi} \cos(x) e^{-inx}\,dx\right]$$

Let $I(n,x) = \int \cos(x) e^{-inx}\,dx$

Then $C_n = (1/2\pi)\left[-(I(n,0) - I(n,-\pi)) + I(n,\pi) - I(n,0)\right]$

$= (1/2\pi)\left[I(n,\pi) + I(n,-\pi) - 2I(n,0)\right]$

For $n \neq \pm 1$:

$$I(n,x) = (1/2)\int (e^{ix} + e^{-ix}) e^{-inx}\,dx = (1/2)\int \left(e^{i(1-n)x} + e^{i(-1-n)x}\right)dx$$

$$= (1/2)\left[\frac{e^{i(1-n)x}}{i(1-n)} + \frac{e^{i(-1-n)x}}{i(-1-n)}\right] = (1/2i)\left[\frac{e^{i(1-n)x}}{1-n} - \frac{e^{i(-1-n)x}}{1+n}\right]$$

$$I(n,\pi) = (1/2i)\left[\frac{(e^{i\pi})^{(1-n)}}{(1-n)} - \frac{(e^{-i\pi})^{(1+n)}}{1+n}\right]$$

$$= (1/2i)\left[\frac{(-1)^{-n}(-1)^{1}}{1-n} + \frac{(-1)(-1)^{1}(-1)^{n}}{1+n}\right]$$

$$= (1/2i)\left[\frac{(-1)^{n}}{n-1} + \frac{(-1)^{n}}{n+1}\right]$$

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$$I(n,-\pi) = (1/2i)\left[\frac{(e^{-i\pi})^{(1-n)}}{1-n} - \frac{(e^{i\pi})^{(1+n)}}{1+n}\right]$$
$$= I(n,\pi) \quad \text{since } e^{i\pi} = e^{-i\pi} = -1$$

$$I(n,0) = (1/2i)\left[\frac{1}{1-n} - \frac{1}{1+n}\right] = (1/2i)\left[\frac{-1}{n-1} - \frac{1}{n+1}\right]$$

So
$$c_n = (1/2\pi)\left[2I(n,\pi) - 2I(n,0)\right]$$
$$= (1/2\pi i)\left[\frac{(-1)^n}{n-1} + \frac{(-1)^n}{n+1} + \frac{1}{n-1} + \frac{1}{n+1}\right]$$
$$= i\left(\frac{-1}{2\pi}\right)\left[\frac{1+(-1)^n}{n-1} + \frac{1+(-1)^n}{n+1}\right]$$

Hence $a_n = 0$, $b_n = \left(\frac{-1}{2\pi}\right)\left[\frac{1+(-1)^n}{n-1} + \frac{1+(-1)^n}{n+1}\right]$ for $n \neq \pm 1$

For $n = 1$:
$$I(1,x) = (1/2)\int(1+e^{-2ix})\,dx = (1/2)(x + e^{-2ix}/(-2i))$$
$$I(1,\pi) = (1/2)(\pi - 1/2i)$$
$$I(1,0) = (1/2)(0 - 1/2i) = -1/4i$$
$$I(1,-\pi) = (1/2)(-\pi - 1/2i)$$

So $c_1 = (1/2\pi)(1/2)(\pi - 1/2i + (-\pi - 1/2i) - 2(-1/4i)) = 0$
and $c_{-1} = c_1^* = 0$

$$\therefore f(x) = \sum_{n=2}^{+\infty} (1/\pi)\left[\frac{1+(-1)^n}{n-1} + \frac{1+(-1)^n}{n+1}\right]\sin(n\omega_0 x)$$
$$= \sum_{n=1}^{+\infty} (1/\pi)\left[\frac{2}{2n-1} + \frac{2}{2n+1}\right]\sin(2nx)$$
$$= \sum_{n=1}^{+\infty} (2/\pi)\left[\frac{4n}{4n^2-1}\right]\sin(2nx)$$
$$= \sum_{n=1}^{+\infty} \left[\frac{8n}{\pi(4n^2-1)}\right]\sin(2nx)$$