TUTORIAL 11 SOLUTIONS (NOVEMBER 16, 2006) – VERSION 2

Fourier Series of a Real Function (Tutorial Problem)

Question: If $f(x)$ is a real function with Fourier series expansion $\sum_{n=-\infty}^{+\infty} C_n e^{inw_0x}$, where $C_n = a_n + ib_n$ and $a_n, b_n \in \mathbb{R}$ for all n, rewrite the Fourier series expansion of $f(x)$ so that it contains only real quantities

Solution:

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{inw_0x}$$

$$= \sum_{n=-\infty}^{-1} C_n e^{inw_0x} + C_0 + \sum_{n=1}^{+\infty} C_n e^{inw_0x} \quad \text{(expand the sum)}$$

$$= C_0 + \sum_{n=1}^{+\infty} \left(C_{-n} e^{i(-n)w_0x} + C_n e^{inw_0x}\right) \quad \text{(change summation index and combine)}$$

$$= a_0 + \sum_{n=1}^{+\infty} \left(C_n^* e^{-inw_0x} + C_n e^{inw_0x}\right) \quad \left(\text{since } f(x) \text{ is real, } C_n = (C_{-n})^*\text{; also } C_0 = C_0^* \Rightarrow C_0 = a_0\right)$$

$$= a_0 + \sum_{n=1}^{+\infty} \left[\left(C_n e^{inw_0x}\right)^* + C_n e^{inw_0x}\right] \quad \left(e^{-nw_0x} = (e^{nw_0x})^*\right)$$

$$= a_0 + \sum_{n=1}^{+\infty} 2 \cdot \mathrm{Re}\left(C_n e^{inw_0x}\right) \quad \left(z + z^* = 2\,\mathrm{Re}(z)\right)$$

$$= a_0 + \sum_{n=1}^{+\infty} 2 \cdot \mathrm{Re}\left[(a_n + ib_n)(\cos(nw_0x) + i\sin(nw_0x))\right] \quad \text{(by Euler's identity)}$$

$$= a_0 + \sum_{n=1}^{+\infty} 2 \cdot \mathrm{Re}\left[a_n\cos(nw_0x) + ia_n\sin(nw_0x) + ib_n\cos(nw_0x) - b_n\sin(nw_0x)\right] \quad \left(\text{multiply terms out, recall } i^2 = -1\right)$$

$$= a_0 + \sum_{n=1}^{+\infty} \left[2a_n\cos(nw_0x) - 2b_n\sin(nw_0x)\right] \quad \text{(take the real part)}$$

$$\therefore \boxed{f(x) = a_0 + \sum_{n=1}^{+\infty} \left[2a_n\cos(nw_0x) - 2b_n\sin(nw_0x)\right]}$$

MATH 264 (Advanced Calculus), Fall 2006, McGill University
Thursday Tutorial, 11:35 A.M. in Wilson 103
TA: Sacha Nandlall (sacha.nandlall@mail.mcgill.ca) http://www.resanova.com/teaching/calculus/

TUTORIAL 11 SOLUTIONS (NOVEMBER 16, 2006) – VERSION 2

<ins>Problem 4 (Parts B and D) and Problem 16</ins>
<ins>(Parts A and D) of section 17.3 (Greenberg)</ins>

Question: Find the Fourier series of the following periodic functions, specified over one period:

4b) $f(x) = |x| \quad , x \in (-\pi, \pi]$

4d) $f(x) = 50 \quad , x \in (0, 2]$

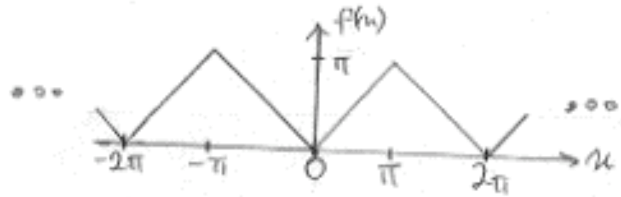
16a) $f(x) = \begin{cases} 50 & \text{if } -1 \le x \le 1 \\ 0 & \text{if } -2 < x < -1 \text{ or } 1 < x \le 2 \end{cases} \quad , x \in (-2, 2]$

16d) $f(x) = 6 \sin x \quad , x \in (0, 2\pi]$

Solution:

4b) $T = \pi - (-\pi) = 2\pi$

$\omega_0 = 2\pi / T = 1$



$$C_n = (1/2\pi) \int_{-\pi}^{\pi} |x| e^{-jn \cdot 1 \cdot x} dx$$

$$= (1/2\pi) \left[\int_{-\pi}^{0} (-x) e^{-jnx} dx + \int_{0}^{\pi} x e^{-jnx} \right]$$

Let $I(n,x) = \int x e^{-jnx} dx$; then

$$C_n = (1/2\pi) \left[-(I(n,0) - I(n,-\pi)) + (I(n,\pi) - I(n,0)) \right]$$

$$= (1/2\pi) \left[I(n,\pi) + I(n,-\pi) - 2 I(n,0) \right]$$

$$I(0,x) = \int x \, dx = x^2/2$$

$$I(n,x)|_{n \neq 0} = x \cdot (-1/in) e^{-inx} - 1 \cdot (-1/in)^2 e^{-inx}$$

$$= \left[i(x/n) + (1/n^2) \right] e^{-inx}$$

u	dv
x	e^{-inx}
1	$(-1/in) e^{-inx}$
0	$(-1/in)^2 e^{-inx}$

$$C_0 = (1/2\pi)\left[\pi^2/2 + (-\pi)^2/2 - 2\cdot 0^2/2\right] = \pi^2/2\pi = \pi/2$$ (again the average value of $f(x)$)

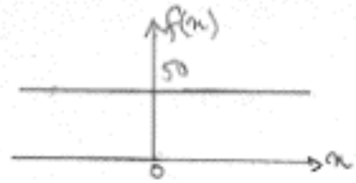
$$C_n\big|_{n\neq 0} = (1/2\pi)\Big[\left(i(\pi/n) + 1/n^2\right)e^{-in\pi} + \left(i(-\pi/n) + 1/n^2\right)e^{in\pi} - 2\left(0 + 1/n^2\right)e^0\Big]$$
$$= (1/2\pi)\left[\left(i\pi/n + 1/n^2 - i\pi/n + 1/n^2\right)(-1)^n - 2/n^2\right]$$
$$= (1/2\pi)\left[(2/n^2)(-1)^n - 2/n^2\right]$$
$$= (1/\pi n^2)\left[(-1)^n - 1\right]$$
$$= \begin{cases} 0 & \text{if } n = 2k \text{ (i.e. } n \text{ is even)} \\ 1/\pi n^2 & \text{if } n = 2k+1 \text{ (i.e. } n \text{ is odd)} \end{cases} \quad (k \in \mathbb{Z})$$

(Check: $C_n = (C_{-n})^*$)

So $a_n = \begin{cases} \pi/2 & \text{if } n = 0 \\ 1/\pi n^2 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$, $b_n = 0$

$$\therefore f(x) = \pi/2 + \sum_{n=1,\, n \text{ odd}}^{+\infty} (2/\pi n^2)\cos(n\omega_0 x)$$
$$= \boxed{\pi/2 + \sum_{n=1}^{+\infty} \left(\frac{2}{\pi(2n-1)^2}\right)\cos((2n-1)x)}$$

4d) We don't need to go through the method of the previous problems:



C_0 is the average value of $f(x)$, so $C_0 = 50$

But then $f(x) = 50 + \sum_{n=1}^{+\infty}(\ldots) \Rightarrow \sum_{n=1}^{+\infty}(\ldots) = f(x) - 50 = 0$

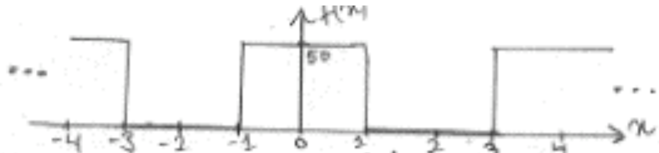
So $\boxed{f(x) = 50}$ is the Fourier series

In general the Fourier series of a constant is the same constant. This isn't surprising, because the function does not vary in time, and so all time-varying terms must be zero.

TUTORIAL 11 SOLUTIONS (NOVEMBER 16, 2006) – VERSION 2

16a) $T = 2-(-2) = 4$

$\omega_0 = 2\pi/T = \pi/2$



$$c_n = (1/4)\int_{-2}^{2} f(x)\, e^{-in(\pi/2)x}\,dx = (1/4)\int_{-1}^{1} 50\, e^{-in(\pi/2)x}\,dx$$
$$= (25/2)\int_{-1}^{1} e^{-in(\pi/2)x}\,dx$$

Let $I(n,x) = \int e^{-in(\pi/2)x}\,dx$; then $c_n = (25/2)\,[I(n,1) - I(n,-1)]$

$I(0,x) = \int 1\cdot dx = x$

$I(n,x)\big|_{n\neq 0} = \left(1/(-in\pi/2)\right) e^{-in(\pi/2)x} = i\left(\frac{2}{n\pi}\right) e^{-in(\pi/2)x}$

So $c_0 = (25/2)\,[1-(-1)] = 50/2 = 25$ (and indeed the average value of $f(x)$ is 25)

$$c_n\big|_{n\neq 0} = (25/2)\left[i(2/n\pi)\, e^{-in\pi/2} - i(2/n\pi)\, e^{in\pi/2}\right]$$
$$= -(25/n\pi)\cdot i\cdot\left[e^{i(n\pi/2)} - e^{-i(n\pi/2)}\right]$$
$$= (50/n\pi)\cdot\left(e^{i(n\pi/2)} - e^{-i(n\pi/2)}\right)/2i$$
$$= (50/n\pi)\,\sin(n\pi/2) \qquad (\text{check: } c_n = (c_{-n})^*)$$

Hence $a_n = \begin{cases} 25 & \text{if } n=0 \\ (50/n\pi)\sin(n\pi/2) & \text{otherwise} \end{cases}$, $b_n = 0$

$$\therefore\ f(x) = 25 + \sum_{n=-\infty,\, n\neq 0}^{+\infty} (50/n\pi)\,\sin(n\pi/2)\, e^{in\pi x/2} \quad (\text{complex form})$$
$$= 25 + \sum_{n=1}^{+\infty} (100/n\pi)\,\sin(n\pi/2)\,\cos(n\pi x/2) \quad (\text{real form})$$

TUTORIAL 11 SOLUTIONS (NOVEMBER 16, 2006) – VERSION 2

16d) $T = 2\pi - 0 = 2\pi$
$\omega_0 = 2\pi/T = 1$

Whenever the function is comprised of sines, cosines, complex exponentials, or a combination of these, there's no need to integrate; instead, use the following trick:

$$f(x) = 6\sin x = 6\left(\frac{e^{ix} - e^{i(-x)}}{2i}\right) = (3/i)e^{ix} - (3/i)e^{-ix}$$
$$= (-3i)e^{ix} + (3i)e^{-ix}$$
$$= (-3i)e^{-i\cdot(-1)\cdot 1\cdot x} + (3i)e^{-i(1)\cdot 1\cdot x}$$
$$= (-3i)e^{-i(-1)\omega_0 x} + (3i)e^{-i(1)\omega_0 x} \quad (\text{since } \omega_0 = 1)$$

By identification with $f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 x}$ we conclude that this is the complex Fourier series for $f(x)$; in other words,
$$c_n = \begin{cases} 3i & \text{if } n = -1 \\ -3i & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \quad (\text{check } c_n = (c_{-n})^*)$$

We could also have found the real Fourier series in a similar way:

$$f(x) = 6\sin(1\cdot 1\cdot x) = 6\sin(1\cdot\omega_0 x) = -2(-3)\sin(1\cdot\omega_0 x)$$

and by identification with
$$f(x) = a_0 + \sum_{n=1}^{+\infty}[(2a_n)\cos(n\omega_0 x) - (2b_n)\sin(n\omega_0 x)]$$

$$a_n = 0, \quad b_n = \begin{cases} -3 & \text{if } n = 1 \\ 3 & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases} \quad (\text{since } b_n = -b_{-n})$$