TA: Sacha Nandlall (sacha.nandlall@mail.mcgill.ca)

TUTORIAL 12 SOLUTIONS (NOVEMBER 23, 2006) – VERSION 1

Problem 3, MATH 264 Assignment 6 (Winter 2006)

Question: Solve the diffusion equation $u_{xx} = u_t$;
$u(0,t) = 0$, $u(10,t) = 100$, $t \in (0, +\infty)$;
$u(x,0) = 0$, $x \in (0,10)$
using the method of separation of variables

Solution:

1) Define the BVP

a) unknown function: $u(x,t)$, temperature

b) Ranges of x and t: $x \in [0,10]$, $t \in [0,+\infty) = \mathbb{R}_+$

c) Governing PDE: heat equation with $\alpha = 1$: $\partial^2 u(x,t)/\partial x^2 = \partial u(x,t)/\partial t$

d) Boundary conditions:

$\rightarrow u(0,t) = 0 \quad \forall t \in \mathbb{R}_+$ (BC1)

$\rightarrow u(10,t) = 100 \quad \forall t \in \mathbb{R}_+$ (BC2)

e) Initial condition:

$\rightarrow u(x,0) = 0 \quad \forall x \in (0,10)$ (IC1)

2) BVP general solution: simply quote this one from memory:

$$u(x,t) = C_1 + C_2 x + \sum_{n=1}^{+\infty} \left[2a_n \cos(n\omega_0 x) - 2b_n \sin(n\omega_0 x)\right] e^{-(n\omega_0)^2 t}$$

where $C_1, C_2, a_n, b_n, \omega_0 \in \mathbb{R}$ and $\omega_0 > 0$ (here $\alpha = 1$)

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3) Apply the boundary and initial conditions

BC1 $u(0,t) = 0 = C_1 + C_2 \cdot 0 + \sum_{n=1}^{+\infty} [a_n\cos(0) - 2b_n \sin(0)] e^{-n^2\omega_0^2 t}$

$= C_1 + \sum_{n=1}^{+\infty} 2a_n e^{-n^2\omega_0^2 t}$

This has to work for all $t \in \mathbb{R}_+$, and so for t "large":

$\lim_{t\to+\infty} u(0,t) = u(0,+\infty) = 0 = C_1 \Rightarrow C_1 = 0$

and for $t=0$:

$u(0,0) = \sum_{n=1}^{+\infty} 2a_n \Rightarrow a_n = 0 \ \forall n$ by linearity

$\therefore$ BC1 shows that $\boxed{C_1 = 0}$ and $\boxed{a_n = 0}$

BC2 $u(10,t) = 100 = C_2 \cdot 10 + \sum_{n=1}^{+\infty} [-2b_n \sin(n\omega_0 \cdot 10)] e^{-n^2\omega_0^2 t}$

Again, this must hold for all $t \in \mathbb{R}_+$, and so for t "large":

$\lim_{t\to+\infty} u(10,t) = u(10,+\infty) = 100 = C_2 \cdot 10 \Rightarrow C_2 = 10$

and for $t=0$:

$u(10,0) = 100 = 10 \cdot 10 + \sum_{n=1}^{+\infty} [-2b_n \sin(n\omega_0 \cdot 10)]$

$\Rightarrow \sum_{n=1}^{+\infty} b_n \sin(10 n\omega_0) = 0$

This time there are two possibilities. The first is that $b_n = 0 \ \forall n$. But if this is the case, the entire transient part of the solution dies, as we've set both $a_n = 0$ and $b_n = 0$; we get

$u(x,t) = 10x$

There is no way this can satisfy IC1, which requires that $u(x,0) = 0 \ \forall x \in (0,10)$

This leaves us only one other possibility:
$\sin(10 n \omega_0) = 0 \ \forall n$, i.e. for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$

This gives $10\omega_0 = \pi \Rightarrow \omega_0 = \pi/10$ (note that ω_0 must be a constant and can't depend on n, so we can't take $\omega_0 = \pi/10n$)

$\therefore$ BC2 shows that $\boxed{C_2 = 10}$ and $\boxed{\omega_0 = \pi/10}$

IC1 $u(x,0) = 0 = 10x + \sum_{n=1}^{+\infty} (-2b_n) \sin(n(\pi/10)x)$

$\Rightarrow -10x = \sum_{n=1}^{+\infty} (-2b_n) \sin(n(\pi/10)x)$

This shows us that if we can expand the left-hand side, $-10x$, into a Fourier sine series with $\omega_0 = \pi/10$, we can find the b_n.

Let $f(x) = -10x$. Its odd extension $f_o(x)$ must have $\omega_0 = \pi/10$ and $T = 2\pi/\omega_0 = 20$; hence take

$$f_o(x) = \begin{cases} -f(-x) & \text{if } -T/2 \le x < 0 \\ f(x) & \text{if } 0 \le x < T/2 \end{cases}$$

$$= \begin{cases} -(-10(-x)) & \text{if } -10 \le x < 0 \\ -10x & \text{if } 0 \le x < 10 \end{cases}$$

$$= -10x \quad \text{for } -10 \le x < 10$$

This function's Fourier series consists only of sines, and it is identical to $f(x)$ for $0 < x < 10$

Thus $b_n = \mathrm{Im}(c_n) = \mathrm{Im}\left((1/T)\int_T f(x) e^{-in\omega_0 x}\right)$

$$= (-1/T)\int_T f(x)\sin(n\omega_0 x)\,dx$$

$$= (1/20)\int_{-10}^{10} (10x)\sin(n\pi x/10)\,dx$$

Let $I(n,x) = \int x \sin(n\pi x/10)\,dx$

$$I(n,x)\big|_{n\neq 0} = -\frac{10x}{n\pi}\cos\left(\frac{n\pi x}{10}\right) + \frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right)$$

(we don't need $I(0,x)$, since $n \neq 0$)

u	dv
x	$\sin(n\pi x/10)$
1	$-\left(\frac{10}{n\pi}\right)\cos(n\pi x/10)$
0	$-\left(\frac{100}{n^2\pi^2}\right)\sin(n\pi x/10)$

$$b_n\big|_{n\neq 0} = (1/2)\left[I(n,10) - I(n,-10)\right]$$

$$= (1/2)\Big[\left((-10\cdot 10/n\pi)\cos(n\pi) + (100/n^2\pi^2)\sin(n\pi)\right) - \left((10\cdot 10/n\pi)\cos(-n\pi) + (100/n^2\pi^2)\sin(-n\pi)\right)\Big]$$

$$= (1/2)\left[2\cdot(-100/n\pi)\cdot(-1)^n\right]$$

$$\therefore \boxed{b_n = -\frac{100(-1)^n}{n\pi}}$$

$$\therefore \boxed{u(x,t) = 10x + \frac{200}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n \sin(n\pi x/10)\,e^{-n^2\pi^2 t/100}}{n}}$$

Problem 4, MATH 264 Assignment 6 (Winter 2006)

Question: Solve the BVP $u_{xx} = u_t$, $u(0,t) = 25$ and $u_x(4,t) = 0$ for $t \in (0,+\infty)$, and $u(x,0) = 25$ for $x \in (0,4)$

Solution:

1) Define the BVP
   a) Unknown function is $u(x,t)$, represents temperature
   b) Ranges of x and t: $x \in (0,4)$, $t \in \mathbb{R}_+ \setminus \{0\} = (0,+\infty)$
   c) PDE: $\partial^2 u(x,t)/\partial x^2 = \partial u(x,t)/\partial t$ (heat eq'n with $\alpha = 1$)
   d) Boundary conditions:
      $\rightarrow u(0,t) = 25 \quad \forall t$ BC1
      $\rightarrow \partial u(x,t)/\partial x\big|_{x=4} = 0 \quad \forall t$ BC2
   e) Initial condition:
      $\rightarrow u(x,0) = 25 \quad \forall x$ IC1

2) General solution: same as previous problem

3) Apply BCs and ICs

BC1 $u(0,t) = 25 = C_1 + \sum_{n=1}^{+\infty} 2a_n e^{-(n\omega_0)^2 t} \quad \forall t$

For $t \to +\infty$ we get $\boxed{C_1 = 25}$ and for $t=0$ we get

$\sum_{n=1}^{+\infty} a_n = 0 \Rightarrow \boxed{a_n = 0 \quad \forall n}$ (by linearity)

BC2 $\partial u/\partial x = C_2 + \sum_{n=1}^{+\infty} -2b_n \cdot n\omega_0 \cdot \cos(n\omega_0 x) e^{-(n\omega_0)^2 t}$

Thus $\partial u/\partial x\big|_{x=4} = 0 = C_2 + \sum_{n=1}^{+\infty} -2b_n \cdot n\omega_0 \cdot \cos(4n\omega_0) e^{-(n\omega_0)^2 t}$

For $t \to +\infty$ we get $\boxed{C_2 = 0}$ and for $t=0$ we get

$\sum_{n=1}^{+\infty} -2b_n n\omega_0 \cdot \cos(4n\omega_0) = 0$

$\Rightarrow -2b_n \cdot n\omega_0 \cdot \cos(4n\omega_0) = 0 \quad \forall n$ (by linearity)

$\Rightarrow b_n \cos(4n\omega_0) = 0 \quad \forall n$ (since $n \neq 0$ and $\omega_0 \neq 0$)

There are two possibilities here.

First we could have $b_n = 0 \ \forall n$

This gives $u(x,t) = 25$, which does indeed satisfy (IC1). So $u(x,t) = 25$ is a valid solution.

The other choice is to take $\cos(4n\omega_0) = 0$

$\Rightarrow (4\omega_0 = \pi/2) \wedge (n \text{ odd}) \Rightarrow (\omega_0 = \pi/8) \wedge (n \text{ odd})$

So to get $b_n \cos(4n\omega_0) = 0$ for all n, take $\omega_0 = \pi/8$ and $b_n = 0$ for n even

This gives $u(x,t) = 25 + \sum_{n=1}^{+\infty} -2b_n \sin(n\pi x/8)\, e^{-(n\pi/8)^2 t}$

Applying (IC1) yields

$u(x,0) = 25 = 25 + \sum_{n=1}^{+\infty} -2b_n \sin(n\pi x/8)$

$\Rightarrow \sum_{n=1}^{+\infty} b_n \sin(n\pi x/8) = 0 \quad \forall x$

$\Rightarrow b_n \sin(n\pi x/8) = 0 \quad \forall x, n$ (by linearity)

$\Rightarrow b_n = 0 \quad \forall n$ (since $\sin(n\pi x/8)$ is not zero for all x)

So we also get $u(x,t) = 25$ in this case

$\therefore$ the solution is $\boxed{u(x,t) = 25}$

This isn't surprising: the initial and final temperatures given are both 25, and the rate of temperature change at $x=4$ is zero.