TUTORIAL 2 SOLUTIONS (SEPTEMBER 21, 2006) – VERSION 1

Problem 11, Section 14.2 (Adams)

Question: Evaluate $\iint_D \ln x \, dA$ where D is the finite region in the first quadrant bounded by the line $2x+2y=5$ and the hyperbola $xy=1$

Solution:

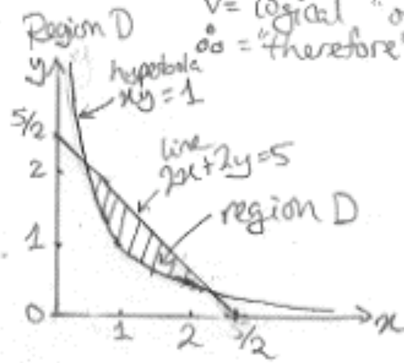
Follow the 6-step method described in the Tutorial 1 Slides

NOTE
$\wedge$ = logical "and"
$\vee$ = logical "or"
$\therefore$ = "therefore"

1) Sketch the region of integration D

→ Line: $2x+2y=5 \Rightarrow y=-x+5/2$
→ Hyperbola: $xy=1 \Rightarrow y=1/x$
→ First quadrant: $(x \geq 0) \wedge (y \geq 0)$

This gives the sketch at the right



2) Choose a coordinate system
Cartesian is fine here

3) Parametrize D in this coordinate system
Let's find the points of intersection of the line and hyperbola:

$(y=-x+5/2) \wedge (y=1/x) \Rightarrow 1/x = -x+5/2 \Rightarrow 1 = -x^2 + (5/2)x$

$\Rightarrow 2x^2 - 5x + 2 = 0$

$$\Rightarrow x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)} = \frac{5}{4} \pm \frac{3}{4} \Rightarrow x \in \{1/2, 2\}$$

$$\therefore D: \begin{cases} 1/2 \leq x \leq 2 \\ \underbrace{1/x}_{\text{hyperbola}} \leq y \leq \underbrace{-x+5/2}_{\text{line}} \end{cases}$$

4) Integrand: $\ln x$

5) Differential element: $dA = dx\,dy$

(→)

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6) Evaluate the integral

$$\iint_D \ln x \, dA = \int_{1/2}^{2} dx \int_{1/x}^{-x+5/2} dy \, \ln x$$
$$= \int_{1/2}^{2} (\ln x) [y]_{1/x}^{-x+5/2} dx$$
$$= \int_{1/2}^{2} (\ln x) [(-x+5/2) - (1/x)] \, dx$$
$$= -\int_{1/2}^{2} (1/x) \ln x \, dx + (5/2) \int_{1/2}^{2} \ln x \, dx - \int_{1/2}^{2} x \ln x \, dx$$

NOTE: We put the y integral inside the x one because its bounds depend on x

Work out the three antiderivatives needed as follows:

$\rightarrow \int (1/x) \ln x \, dx = \int u \, du$
$= u^2/2 = [\ln(x)]^2/2$

Let $u = \ln x$. Then $du = (1/x)\,dx$ (by substitution)

$\rightarrow \int \ln x \, dx = x \ln x - \int x (1/x) \, dx$
$= x \ln x - \int dx = x \ln x - x$

u	dv
$\ln x$	dx
$(1/x)\,dx$	x

(by parts)

$\rightarrow \int x \ln x \, dx = x \int \ln x \, dx - \int [\int \ln x \, dx] \, dx$
$= x[x \ln x - x] - \int [x \ln x - x] \, dx$
(replace $\int \ln x \, dx$ by the antiderivative we found above)
$= x^2 \ln x - x^2 - \int x \ln x \, dx + \int x \, dx$
$= x^2 \ln x - x^2 - \int x \ln x \, dx + x^2/2$
$= x^2 [\ln x - 1/2] - \int x \ln x \, dx$

u	dv
x	$\ln x \, dx$
dx	$\int \ln x \, dx$

(by parts)

Solve for $\int x \ln x \, dx$:

$$2 \int x \ln x \, dx = x^2[\ln x - 1/2] \Rightarrow \int x \ln x \, dx = (x^2/2)[\ln x - 1/2]$$

Thus,

$$\iint_D \ln x \, dA = -\left[(\ln 2)^2/2 - (\ln(1/2))^2/2\right] + (5/2)\left[[2\ln 2 - 2] - [\tfrac{1}{2}\ln(1/2) - 1/2]\right]$$
$$- \left[[(2^2)(1/2)[\ln 2 - 1/2]] - [(1/2)^2(1/2)[\ln(1/2) - 1/2]]\right]$$
$$= -(1/2)[(\ln 2)^2 - (-\ln 2)^2] + (5/2)[2\ln 2 + (1/2)\ln 2 - 2 + 1/2]$$
$$- [2\ln 2 - 1 + (1/8)\ln 2 + 1/16]$$
$$= 0 + (25/4)\ln 2 - 15/4 - [(17/8)\ln 2 - 15/16]$$
$$= (33/8)\ln 2 - (45/16)$$

NOTE: $\ln(x^a) = a \ln x$
So $\ln(1/2) = \ln(2^{-1}) = -\ln 2$

$$\therefore \iint_D \ln x \, dA = \boxed{(33/8)\ln 2 - (45/16)}$$

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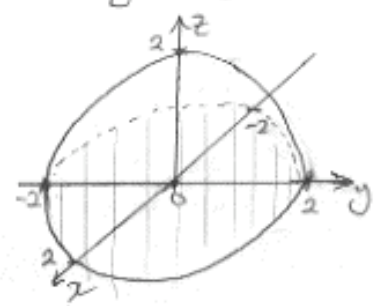
Problem 3, Section 14.5 (Adams)

Question: Evaluate $\iiint_D (3+2xy)\,dV$ where D is the solid hemispherical dome given by $x^2+y^2+z^2 \le 4$ and $z \ge 0$

Solution:

As always, follow the same 6 steps

Region D



1) Sketch the region
This is the top half of the hemisphere bounded by $x^2+y^2+z^2 = 2^2$ (centered at origin, radius 2)

2) Coordinate system: we have a sphere, so use spherical

3) Parametrize D: easily done in spherical: $\begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/2 \end{cases}$

4) Integrand: $3+2xy = 3+2(\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)$
$= 3+2\rho^2\sin^2\phi\sin\theta\cos\theta$

5) $dV = (\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$ (Jacobian determinant for spherical is $\rho^2\sin\phi$)

6) Evaluate the integral

$$\iiint_D (3+2xy)\,dV = \int_0^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi\,(\rho^2\sin\phi)(3+2\rho^2\sin^2\phi\sin\theta\cos\theta)$$
$$= 3\int_0^2 \rho^2\,d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2}\sin\phi\,d\phi + 2\int_0^2 \rho^4\,d\rho \int_0^{2\pi}\sin\theta\cos\theta\,d\theta \int_0^{\pi/2}\sin^3\phi\,d\phi$$

Now $\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = \int_0^{2\pi} u\,du = \left[u^2/2\right]_0^{2\pi}$ $\longrightarrow$ Let $u = \sin\theta$ Then $du = \cos\theta\,d\theta$
$= [\sin(2\pi)]^2/2 - [\sin(0)]^2/2 = 0$

The second term disappears and we get

$$\iiint_D (3+2xy)\,dV = 3\left[\rho^3/3\right]_0^2 [\theta]_0^{2\pi} [-\cos\phi]_0^{\pi/2} + 0$$
$$= (2^3-0)(2\pi-0)(-\cos(\pi/2)+\cos(0))$$
$$= 16\pi$$

$\therefore \iiint_D (3+2xy)\,dV = \boxed{16\pi}$

(check with Maple)

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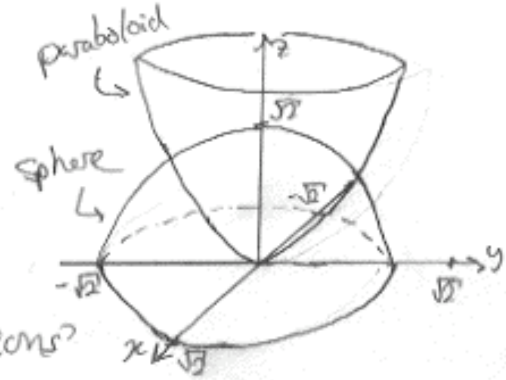
Problem 29, Section 14.6 (Adams)

Question: Find $\iiint_R z\,dV$ over the region R satisfying $x^2+y^2 \le z \le \sqrt{2-x^2-y^2}$

Solution:

1) Sketch R



$\to z = x^2+y^2$ is a paraboloid

$\to z = \sqrt{2-x^2-y^2} \Leftrightarrow (x^2+y^2+z^2 = (\sqrt{2})^2) \wedge (z \ge 0)$ is the top half of a sphere of radius $\sqrt{2}$ centered at the origin

2) Choose a coordinate system:

owing to the "x^2+y^2" terms in both regions' descriptions, we pick Cylindrical (Spherical may also be a valid choice however)

3) Parametrize R

Same technique as in Problem 19, Section 14.6 (Adams) of Tutorial 1

$\to z$ varies between the paraboloid (bottom) and sphere (top)

$\to$ we can get r and θ by projecting the volume onto the xy plane

Parametrize the surface equations:

$\to$ Paraboloid: $z = x^2+y^2 = r^2 \Rightarrow z = r^2$

$\to$ Sphere: $z = \sqrt{2-(x^2+y^2)} = \sqrt{2-r^2} \Rightarrow z = \sqrt{2-r^2}$

Thus $r^2 \le z \le \sqrt{2-r^2}$

From the sketch, we see that the xy plane projection of the volume R will be the surface bounded by the curve of intersection of the paraboloid and sphere. (Convince yourself that this is true by imagining looking at the 3D sketch from the top)

The equation for this curve of intersection is given by:

$(z = r^2) \wedge (z = \sqrt{2-r^2}) \Rightarrow r^2 = \sqrt{2-r^2} \Rightarrow (r^2)^2 = 2-r^2$

$\Rightarrow (r^2)^2 + (r^2) - 2 = 0 \Rightarrow (r^2-1)(r^2+2) = 0$

$\Rightarrow r^2 = 1 \Rightarrow r = 1$ (since $r \ge 0$ and $r \in \mathbb{R}$)

This is a circle of radius 1, and so $0 \le r \le 1$ and $0 \le \theta \le 2\pi$

$$\therefore R: \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ r^2 \le z \le \sqrt{2-r^2} \end{cases}$$

$(\longrightarrow)$

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4) Integrand : z

5) $dV = r\,dr\,d\theta\,dz$ (the Jacobian determinant for cylindrical is r)

6) Evaluate the integral

$$\begin{aligned}
\iiint_R z\,dV &= \int_0^1 dr \int_0^{2\pi} d\theta \int_{r^2}^{\sqrt{2-r^2}} dz\,(r)(z) \\
&= \int_0^{2\pi} d\theta \int_0^1 r\left[\int_{r^2}^{\sqrt{2-r^2}} z\,dz\right] dr \\
&= 2\pi \int_0^1 r\left[z^2/2\right]_{r^2}^{\sqrt{2-r^2}} dr \\
&= \pi \int_0^1 r\left((2-r^2) - r^4\right) dr \\
&= \pi \int_0^1 \left[2r - r^3 - r^5\right] dr \\
&= \pi\left[r^2 - r^4/4 - r^6/6\right]_0^1 \\
&= \pi\left[(1 - 1/4 - 1/6) - 0\right] \\
&= \pi\left(12/12 - 3/12 - 2/12\right) \\
&= 7\pi/12
\end{aligned}$$

$$\therefore \iiint_R z\,dV = \boxed{7\pi/12}$$ (check with Maple)