TUTORIAL 5 SOLUTIONS (OCTOBER 12, 2006) – VERSION 1

Problem 3, Section 15.3 (Adams)

Question: Find the mass of a wire along the curve

$\vec{r}(t) = (3t)\hat{\imath} + (3t^2)\hat{\jmath} + (2t^3)\hat{k}$, $0 \le t \le 1$

if the density at $\vec{r}(t)$ is $\delta(t) = (1+t)$ grams/unit length

Solution:

Denoting the wire's curve as C, we want the line integral of density along C, i.e. $\int_C \delta \, ds$; use the 5-step method in Tutorial 5:

1) Place the integral in standard scalar or vector form: already done

2) Parametrize the curve C

The line integral is scalar (no direction of integration is specified), and so we want a parametrization of the form $C \sim \vec{r}(t)$, $a \le t \le b$

This is already given in the question

3) Express the integrand in terms of t: done already ($\delta = 1+t$)

4) Write an expression for the differential element

$$d\vec{r} = (d\vec{r}(t)/dt)\,dt = [3\hat{\imath} + (6t)\hat{\jmath} + (6t^2)\hat{k}]\,dt$$

$$ds = |d\vec{r}| = \sqrt{3^2 + (6t)^2 + (6t^2)^2}\,dt = \sqrt{36t^4 + 36t^2 + 9}\,dt$$

$$= \sqrt{36}\sqrt{t^4 + t^2 + 1/4}\,dt = 6\sqrt{(t^2 + 1/2)^2}\,dt$$

$$= (6t^2 + 3)\,dt$$

5) Evaluate the integral:

$$\int_C \delta\, ds = \int_0^1 (1+t)(6t^2+3)\,dt = \int_0^1 [6t^3 + 6t^2 + 3t + 3]\,dt$$

$$= [6t^4/4 + 6t^3/3 + 3t^2/2 + 3t]_0^1$$

$$= [6/4 + 6/3 + 3/2 + 3]$$

$$= 8$$

$\therefore$ the wire has a mass of $\boxed{8 \text{ grams}}$

Problem 7, Section 15.3 (Adams)

Question: Find $\int_C x^2\,ds$ along the line of intersection of the planes $x-y+z=0$ and $x+y+2z=0$, from the origin to the point $(3,1,-2)$

Solution:

Same 5 steps as the previous problem.

1) $\int_C x^2\,ds$ is in standard scalar form
2) The line integral is vector (key words: from point X to point Y)
   Thus we want a parametrization of the form
   $C \sim \vec{r}(t)$, t from a to b (with $a<b$ or $a>b$)
   For intersections of surfaces, follow the tips in the Tutorial 5 slides
   First get one equation in two variables.
   $(x-y+z=0) \wedge (x+y+2z=0) \Rightarrow (x=y-z) \wedge (x=-y-2z)$
   $\Rightarrow y-z=-y-2z \Rightarrow z=-2y$
   Let's (arbitrarily) pick $y=t$; then $z=-2t$ and $x=t-(-2t)=3t$
   y goes from 0 to 1 so t goes from 0 to 1
   $\therefore$ $C \sim \vec{r}(t) = (3t)\hat{\imath} + t\hat{\jmath} - (2t)\hat{k}$, t from 0 to 1

3) Integrand: $x^2 = (3t)^2 = 9t^2$
4) $d\vec{r} = (d\vec{r}(t)/dt)\,dt = [3\hat{\imath} + \hat{\jmath} - 2\hat{k}]\,dt$
   $ds = |d\vec{r}| = \sqrt{3^2+1^2+2^2}\,dt = \sqrt{14}\,dt$

5) $\int_C x^2\,ds = \int_0^1 9t^2 \cdot \sqrt{14}\,dt = 9\sqrt{14}\,[t^3/3]_0^1$
   $= 3\sqrt{14}$

$$\therefore \int_C x^2\,ds = \boxed{3\sqrt{14}}$$

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Problem 1, Section 15.4 (Adams)

Question: Evaluate the line integral of the tangential component of vector field $\vec{F} = (xy)\hat{\imath} - x^2\hat{\jmath}$ along $y = x^2$ from $(0,0)$ to $(1,1)$

Solution:

We want $\int_C \vec{F}\cdot d\vec{r}$; follow the 5-step method

1) The integral is in standard vector form
2) Parametrize the curve C
   This is a vector line integral (direction of integration specified)
   We easily get $C \sim \vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath}$, t from 0 to 1
   where we let $x = t$ (and hence $y = x^2 = t^2$)
3) Integrand: $\vec{F} = (t\cdot t^2)\hat{\imath} - (t)^2\hat{\jmath} = t^3\hat{\imath} - t^2\hat{\jmath}$
4) $d\vec{r} = (d\vec{r}(t)/dt)\,dt = [\hat{\imath} + (2t)\hat{\jmath}]\,dt$
5) Evaluate the integral

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t^3\hat{\imath} - t^2\hat{\jmath})\cdot(\hat{\imath} + (2t)\hat{\jmath})\,dt$$
$$= \int_0^1 (t^3 - 2t\cdot t^2)\,dt = \int_0^1 (-t^3)\,dt$$
$$= -\left[t^4/4\right]_0^1$$
$$= -1/4$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \boxed{-1/4}$$

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Problem #5, Section 15.4 (Adams)

Question: Evaluate the line integral of the tangential component of vector field $\vec{F} = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$ from $(-1,0,0)$ to $(1,0,0)$ in either direction of the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $z=y$.

Solution:

Since changing direction only changes the sign of the answer and the problem says we can pick any direction, we'd expect an answer of zero ($+x=-x$ only if $x=0$). Let's see if this is true:

1) $\int_C \vec{F}\cdot d\vec{r}$ is in standard vector form

2) Parametrize C:

You can either view this as a scalar line integral (direction unspecified), or pick a direction and view it as a vector one. We'll interpret it as scalar here.

The $x^2+y^2=1$ expression suggests $x=\cos t$, $y=\sin t$. Then $z=y=\sin t$.

$x=1 \Rightarrow \cos t = 1 \Rightarrow t = 2\pi k_1$ ($k_1\in\mathbb{Z}$, i.e. k_1 is an integer)

$x=-1 \Rightarrow \cos t = -1 \Rightarrow t = \pi + 2\pi k_2$ ($k_2\in\mathbb{Z}$)

Any values for k_1 and k_2 will do; take $k_1=k_2=0 \Rightarrow t=0$ and $t=\pi$

$\therefore C \sim \vec{r}(t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath} + (\sin t)\hat{k}$, $0\le t\le\pi$

3) Integrand: $\vec{F} = (\sin t\cdot\sin t)\hat{\imath} + (\cos t\cdot\sin t)\hat{\jmath} + (\cos t\cdot\sin t)\hat{k}$

$= (\sin^2 t)\hat{\imath} + (\sin t\cos t)\hat{\jmath} + (\sin t\cos t)\hat{k}$

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4) Differential element

$$d\vec{r} = \left(d\vec{r}(t)/dt\right)dt = [(-\sin t)\hat{\imath} + (\cos t)\hat{\jmath} + (\cos t)\hat{k}]\,dt$$

5) Evaluate the integral

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^\pi [(\sin^2 t)(-\sin t) + (\sin t\cos t)(\cos t) + (\sin t\cos t)(\cos t)]\,dt$$
$$= \int_0^\pi [-\sin^3 t + 2\sin t\cos^2 t]\,dt$$

Now $\int \sin^3 t\,dt = \int (1-\cos^2 t)\sin t\,dt$ Let $u = \cos t$

$= \int (1-u^2)(-du)$ Then $du = -\sin t\,dt$ $\Rightarrow \sin t\,dt = -du$

$= u^3/3 - u$

$= (\cos^3 t)/3 - \cos t$

and $\int \sin t\cos^2 t\,dt = \int u^2(-du) = -u^3/3 = -(\cos^3 t)/3$

$$\therefore \int \vec{F}\cdot d\vec{r} = -\left[(\cos^3 t)/3 - \cos t\right]_0^\pi + 2\left[-(\cos^3 t)/3\right]_0^\pi$$
$$= -\left[\left[(-1)^3/3 - (-1)\right] - \left[1^3/3 - 1\right]\right] + 2\left[-(-1)^3/3 + 1/3\right]$$
$$= -(-1/3 + 1 - 1/3 + 1) + 2(2/3)$$
$$= -4/3 + 4/3$$
$$= 0$$

$\therefore \int \vec{F}\cdot d\vec{r} = \boxed{0}$ as expected

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Problem #17, Section 15.4 (Adams)

Question: Evaluate a) $\oint_C x\,dy$, and b) $\oint_C y\,dx$, counterclockwise along the boundary of the half-disk $x^2+y^2 \leq a^2$, $y \geq 0$

Solution:
Use the 5-step method and do both integrals in parallel

1) We'll place both integrals in standard vector form:

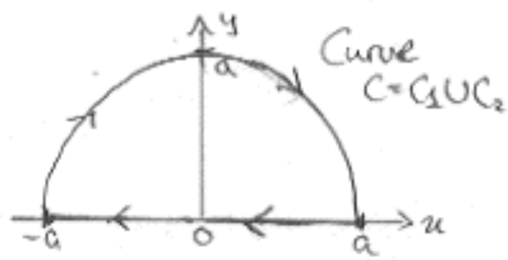
a) $\oint_C x\,dy = \oint_C (x\hat{j}) \cdot d\vec{r}$

b) $\oint_C y\,dx = \oint_C (y\hat{i}) \cdot d\vec{r}$

where we've used the fact that $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

2) Parametrize the curve C

It helps to sketch the curve here.



Let C_1 be the part along $y=0$,
and C_2 be the part along the arc
(thus, $C = C_1 \cup C_2$ (union of C_1 and C_2))

Parametrize C_1 and C_2 separately:

C_1: $\vec{r}_1(t_1) = t_1\hat{i}$, t_1 from $+a$ to $-a$

C_2: $\vec{r}_2(t_2) = (a\cos t_2)\hat{i} + (a\sin t_2)\hat{j}$, t_2 from π to 0

Note that the line integrals are vector since the direction of integration is specified.

Also, since only a^2 appears in the problem, we can take $a \geq 0$ without loss of generality.

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3) Integrands:

Let $\vec{F}_a = x\hat{\jmath}$ (from part a) and $\vec{F}_b = y\hat{\imath}$ (from part B)
Each of these integrands must be evaluated along both C_1 and C_2:

a) Along C_1, $\vec{F}_a = t_1\hat{\jmath}$
Along C_2, $\vec{F}_a = (a\cos t_2)\hat{\jmath}$

b) Along C_1, $\vec{F}_b = \vec{0}$ (since $y=0$ on C_1)
Along C_2, $\vec{F}_b = (a\sin t_2)\hat{\imath}$

4) Differential elements:

For C_1, $d\vec{r}_1 = (d\vec{r}_1(t_1)/dt_1)\,dt_1 = (\hat{\imath})\,dt_1$
For C_2, $d\vec{r}_2 = [(-a\sin t_2)\hat{\imath} + (a\cos t_2)\hat{\jmath}]\,dt_2$

5) Evaluate the integrals:

a)
$$\oint_C x\,dy = \oint_C \vec{F}_a\cdot d\vec{r} = \int_{C_1}\vec{F}_a\cdot d\vec{r}_1 + \int_{C_2}\vec{F}_a\cdot d\vec{r}_2$$
$$= \int_a^{-a}(t_1\hat{\jmath})\cdot(\hat{\imath})\,dt_1 + \int_\pi^0 [(a\cos t_2)\hat{\jmath}]\cdot d\vec{r}_2$$
$$= 0 + \int_\pi^0 a^2\cos^2 t_2\,dt_2$$
$$= (a^2/2)\int_\pi^0 (1+\cos(2t_2))\,dt_2$$
$$= (a^2/2)[[0-\pi]+0]$$
$$= -\pi a^2/2$$

→ Note: $\cos(2t_2)$ has period π, so integrating it from π to 0 (one period) gives zero

b)
$$\oint_C y\,dx = \int_{C_1}\vec{F}_b\cdot d\vec{r}_1 + \int_{C_2}\vec{F}_b\cdot d\vec{r}_2$$
$$= 0 + \int_\pi^0(-a^2\sin^2 t_2)\,dt_2$$
$$= (a^2/2)\int_0^\pi (1-\cos(2t_2))\,dt_2$$
$$= \pi a^2/2$$

$$\therefore \oint_C x\,dy = \boxed{-\pi a^2/2} \quad\text{and}\quad \oint_C y\,dx = \boxed{\pi a^2/2}$$