TUTORIAL 8 SOLUTIONS (NOVEMBER 2, 2006) – VERSION 1

Problem 3, Section 16.3 (Adams)

Question: Evaluate $\oint_C [(x\sin(y^2)-y^2)\,dx + (x^2 y\cos(y^2)+3x)\,dy]$ where C is the CCW boundary of the trapezoid with vertices $(0,-2)$, $(1,-1)$, $(1,1)$, and $(0,2)$

Solution:

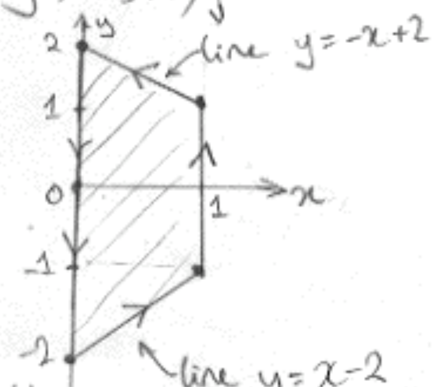
First rewrite the integral as $\oint_C \vec{F}\cdot d\vec{s}$

where $\vec{F} = (x\sin(y^2)-y^2)\hat{\imath} + (x^2 y\cos(y^2)+3x)\hat{\jmath}$

Let S be the surface bounded by C as shown in the sketch; then

$$\oint_C \vec{F}\cdot d\vec{s} = \iint_S (\mathrm{curl}\,\vec{F})\cdot d\vec{S}$$

where $d\vec{S}$ points in the $+\hat{k}$ direction (out of the page), i.e. $d\vec{S} = \hat{k}\,dS$



Now evaluate the surface integral:

1) Sketch the surface: see the sketch on the right

2) Coordinate system: Cartesian

3) Parametrize the surface: $\vec{r}(x,y) = x\hat{\imath}+y\hat{\jmath}$, where $\begin{cases} 0\le x\le 1 \\ x-2\le y\le -x+2 \end{cases}$

4) Integrand: $\mathrm{curl}\,\vec{F} = \left(\partial F_2/\partial x - \partial F_1/\partial y\right)\hat{k}$

$$= [2xy\cos(y^2)+3-(x\sin(y^2)\cdot 2y-2y)]\hat{k}$$
$$= (2y+3)\hat{k}$$

5) $d\vec{S} = (+\hat{k})\,dS = \hat{k}\,dx\,dy$

6) Evaluate the integral

$$\iint_S (\mathrm{curl}\,\vec{F})\cdot d\vec{S} = \int_0^1 dx \int_{x-2}^{-x+2} dy\,(2y+3)(1)$$
$$= 2\int_0^1 dx\int_{x-2}^{-x+2} y\,dy + 3\int_0^1 dx\int_{x-2}^{-x+2} dy$$
$$= \int_0^1 [(-x+2)^2-(x-2)^2]\,dx + 3\int_0^1 [(-x+2)-(x-2)]\,dx$$
$$= \int_0^1 [0]\,dx + 3\int_0^1 [-2x+4]\,dx = 3\left[-x^2+4x\right]_0^1 = 9$$

$$\therefore \oint_C [(x\sin(y^2)-y^2)\,dx + (x^2 y\cos(y^2)+3x)\,dy] = \boxed{9}$$

Problem 5, Section 16.3 (Adams)

Question: Using a line integral, find the plane area enclosed by the curve $\vec{r}(t) = a\cos^3 t\,\hat{\imath} + b\sin^3 t\,\hat{\jmath}$, $0 \le t \le 2\pi$

Solution:

We want to find $\iint_S dS$. If we could find a planar function $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath}$ such that $dS = (\text{curl}\,\vec{F}) \cdot d\vec{S}$, we could apply Stokes' theorem to compute $\iint_S dS$ (where S is the surface enclosed by C, with $d\vec{S}$ oriented such that C is the positively-oriented boundary of S)

Take t going from 0 to 2π, i.e. C oriented CCW. Then $d\vec{S} = +\hat{k}\,dS$.

Now $\text{curl}\,\vec{F} = \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right)\hat{k}$

Thus we want

$$dS = \text{curl}\,\vec{F} \cdot d\vec{S} \Rightarrow dS = \left(\partial F_2/\partial y - \partial F_1/\partial x\right)\hat{k} \cdot \hat{k}\,dS$$

$$\Rightarrow \partial F_2/\partial y - \partial F_1/\partial x = 1 \Rightarrow \partial F_2/\partial y = \partial F_1/\partial x + 1$$

$$\Rightarrow F_2 = \int (\partial F_1/\partial x + 1)\,dx + k(y)$$

where $k(y)$ is an arbitrary function of y.

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Since this is the only equation $\vec{F}$ must satisfy, we are free to choose F_1 and $k(y)$ as we wish; let's pick $F_1 = k(y) = 0$. Then

$$F_2 = \int (0+1)\,dx + 0 = x$$

So our choice for $\vec{F}$ is $\vec{F} = x\hat{j}$ $(F_1 = 0, F_2 = x)$

As a check, $\text{curl}\,\vec{F}\cdot d\vec{S} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = (1\cdot\hat{k})\cdot(\hat{k}\,dS) = dS$ (as required)

Thus $\iint_S dS = \iint_S (\text{curl}\,\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$; evaluate the line integral

1) Place in standard scalar or vector form: the integral is in standard vector form

2) Parametrize the curve:

$$C \sim \vec{r}(t) = a\cos^3 t\,\hat{i} + b\sin^3 t\,\hat{j}, \quad t \text{ from } 0 \text{ to } 2\pi$$

(note: we picked t from 0 to 2π, and not 2π to 0, because we chose $d\vec{S} = +\hat{k}\,dS$, and Stokes' theorem requires that C be oriented positively, i.e. CCW in this case)

3) $\vec{F} = x\hat{j} = a\cos^3 t\,\hat{j}$

4) $d\vec{r} = (d\vec{r}(t)/dt)\,dt = (3a\cos^2 t\sin t\,\hat{i} + 3b\sin^2 t\cos t\,\hat{j})\,dt$

5) $\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (3ab\sin^2 t\cos^4 t)\,dt = 3ab\int_0^{2\pi} (1-\cos^2 t)\cos^4 t\,dt$

$$= 3ab\int_0^{2\pi} (\cos^4 t - \cos^6 t) = 3ab\int_0^{2\pi}\left[(\cos^2 t)^2 - (\cos^2 t)^3\right]dt$$

$$= 3ab\int_0^{2\pi}\left[\left(\frac{1+\cos 2t}{2}\right)^2 - \left(\frac{1+\cos 2t}{2}\right)^3\right]dt$$

$$= (3ab/4)\int_0^{2\pi}\left[1 + 2\cos(2t) + \cos^2(2t)\right]dt - (3ab/8)\int_0^{2\pi}\left[1 + 3\cos(2t) + 3\cos^2(2t) + \cos^3(2t)\right]dt$$

$$= (3ab/4)\int_0^{2\pi}\left[1 + \cos^2(2t)\right]dt - (3ab/8)\int_0^{2\pi}\left[1 + 3\cos^2(2t)\right]dt$$

$$= (3ab/4)\int_0^{2\pi}\left[1 + (1/2)[1+\cos(4t)]\right]dt - (3ab/8)\int_0^{2\pi}\left[1 + (3/2)[1+\cos(4t)]\right]dt$$

$$= (3ab/4)\int_0^{2\pi}(3/2)\,dt - (3ab/8)\int_0^{2\pi}(5/2)\,dt = (3ab/4)\cdot 2\pi\cdot[3/2 - (1/2)(5/2)]$$

$$= 3\pi ab/8$$

$\therefore$ the area enclosed by the curve is $\boxed{3\pi ab/8 \text{ units}^2}$

TUTORIAL 8 SOLUTIONS (NOVEMBER 2, 2006) – VERSION 1

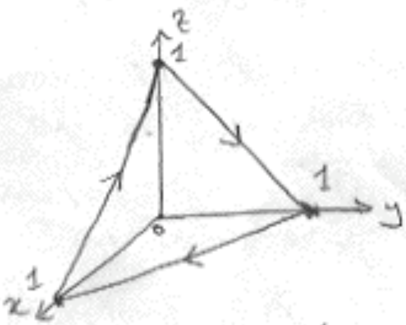
Problem 1, Section 16.5 (Adams)

Question: Evaluate $\oint_C [xy\,dx + yz\,dy + zx\,dz]$ around the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented CW as seen from the point $(1,1,1)$

Solution:

Rewrite the line integral as $\oint_C \vec{F}\cdot d\vec{r}$ where
$\vec{F} = xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$

The curve C is sketched at the right



Let S be the surface bounded by C

Then by Stokes' theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\mathrm{curl}(\vec{F}))\cdot d\vec{S}$$

where $d\vec{S}$ is oriented according to the right-hand rule described in the tutorial slides.

Now evaluate the surface integral (see Tutorials 5 and 6):

1) Sketch the region: see above
2) Pick a coordinate system: Cartesian (we have a plane)
3) Parametrize the surface

we'll need an equation for the plane, of the form
$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$
where $a\hat{\imath}+b\hat{\jmath}+c\hat{k}$ is a normal to the plane
and (x_0,y_0,z_0) is any point in the plane

Let's pick $(x_0,y_0,z_0) = (1,0,0)$ (any other point in the plane would also do just as well)

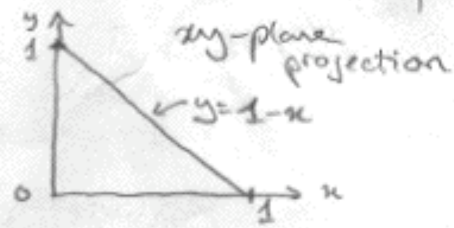
We could find a, b, and c by picking three other points in the plane and setting up three equations to solve for a, b, and c.

However, because the plane is tilted by $45°$ with respect to the coordinate planes $x=0$, $y=0$, and $z=0$, a normal to the plane is $\hat{\imath}+\hat{\jmath}+\hat{k}$ by inspection

Thus $a=1, b=1$, and $c=1$, and an equation for the plane is $x-1+y+z=0$

Solving for z: $z = 1-x-y$ (we could have solved for x and y too; as long as we only have two variables left at the end)

We now need to find bounds on both x and y.
If we project the region into the xy plane, we get the sketch shown at the right



So $0 \le x \le 1$ and $0 \le y \le 1-x$

$\therefore \vec{r}_S(x,y) = x\hat{\imath} + y\hat{\jmath} + (1-x-y)\hat{k}$, $(0 \le x \le 1) \wedge (0 \le y \le 1-x)$

4) Integrand:

$$\operatorname{curl}\vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{bmatrix} = \hat{\imath}[0-y] - \hat{\jmath}[z-0] + \hat{k}[0-x]$$
$$= -y\hat{\imath} - z\hat{\jmath} - x\hat{k}$$
$$= -y\hat{\imath} - (1-x-y)\hat{\jmath} - x\hat{k}$$

(don't forget, we parametrized in terms of x and y, so we have to substitute for z!)

5) Find $d\vec{S}$

$\partial\vec{r}_S/\partial x = \hat{\imath} - \hat{k}$ $\qquad \partial\vec{r}_S/\partial y = \hat{\jmath} - \hat{k}$

$$d\vec{S} = \pm\left(\partial\vec{r}_S/\partial x \times \partial\vec{r}_S/\partial y\right)dx\,dy = \pm\left((\hat{\imath}-\hat{k})\times(\hat{\jmath}-\hat{k})\right)dx\,dy$$
$$= \pm\left((\hat{\imath}-\hat{k})\times\hat{\jmath} + (\hat{\imath}-\hat{k})\times(-\hat{k})\right)dx\,dy \qquad \hat{\imath}\to\hat{\jmath}\to\hat{k}$$
$$= \pm\left(\hat{k} + \hat{\imath} + \hat{\jmath} + \vec{0}\right)dx\,dy = \pm(\hat{\imath}+\hat{\jmath}+\hat{k})\,dx\,dy$$

Now $d\vec{S}$ must be oriented according to the right hand rule. From the sketch on the previous page, $d\vec{S}$ has to point downward, so its $\hat{k}$ component must be negative

$\therefore d\vec{S} = -(\hat{\imath}+\hat{\jmath}+\hat{k})\,dx\,dy$

6) Evaluate the integral

$$\iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S} = \int_0^1 dx \int_0^{1-x} dy\,\left(-y\hat{\imath} - (1-x-y)\hat{\jmath} - x\hat{k}\right)\cdot\left(-(\hat{\imath}+\hat{\jmath}+\hat{k})\right)$$
$$= \int_0^1 dx \int_0^{1-x} [y + (1-x-y) + x]\,dy = \int_0^1 dx \int_0^{1-x} dy$$
$$= \int_0^1 (1-x-0)\,dx = \left[x - x^2/2\right]_0^1 = (1 - 1^2/2) - (0-0) = 1/2$$

$$\therefore \oint_C [xy\,dx + yz\,dy + zx\,dz] = \boxed{1/2}$$

TUTORIAL 8 SOLUTIONS (NOVEMBER 2, 2006) – VERSION 1

Problem 3, Section 16.5 (Adams)

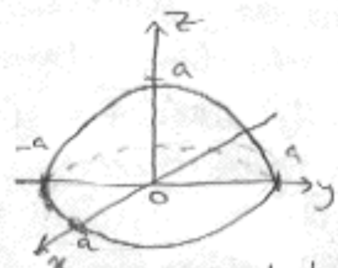
Question: Evaluate $\iint_S (\text{curl}(\vec{F})\cdot\hat{N}\,dS$ where S is the hemisphere $x^2+y^2+z^2=a^2$, $z\geq 0$, with outward normal, and $\vec{F}=3y\hat{i}-2xz\hat{j}+(x^2-y^2)\hat{k}$

Solution:

Let C be the curve bounding the hemisphere, oriented according to the right-hand rule; then

$\iint_S (\text{curl}(\vec{F})\cdot\hat{N}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

by Stokes' theorem

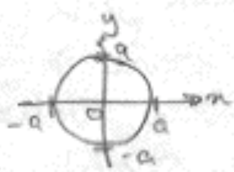


(assume without loss of generality that $a \geq 0$)

Evaluate the line integral as follows

1) Parametrize the curve C

We see that the curve C bounding the surface is just the circle $x^2+y^2=a^2$, so use the parametrization $x=a\cos t$, $y=a\sin t$, $z=0$



$\hat{N}$, and thus $d\vec{S}$, point out of the surface, and so by the right-hand rule C must be oriented CCW

Thus we let t go from 0 to 2π

$\therefore C \sim \vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$, t from 0 to 2π

2) $d\vec{r} = (d\vec{r}(t)/dt)\,dt = (-a\sin t\,\hat{i} + a\cos t\,\hat{j})\,dt$

3) $\vec{F} = 3a\sin t\,\hat{i} - a^2(\cos^2 t - \sin^2 t)\hat{k}$

4) $\vec{F}\cdot d\vec{r}(t)/dt = -3a^2\sin^2 t$

5) $\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}[\vec{F}\cdot d\vec{r}(t)/dt]\,dt = -3a^2\int_0^{2\pi}\sin^2(t)\,dt$

$= -3a^2\int_0^{2\pi}\left[\dfrac{1-\cos(2t)}{2}\right]dt =$

$= (-3a^2/2)\left[\int_0^{2\pi}dt - \underbrace{\int_0^{2\pi}\cos(2t)\,dt}_{\to 0}\right]$

$= (-3a^2/2)(2\pi+0) = -3\pi a^2$

$\therefore \iint_S (\text{curl}(\vec{F})\cdot\hat{N}\,dS = \boxed{-3\pi a^2}$

Problem 5, Section 16.5 (Adams)

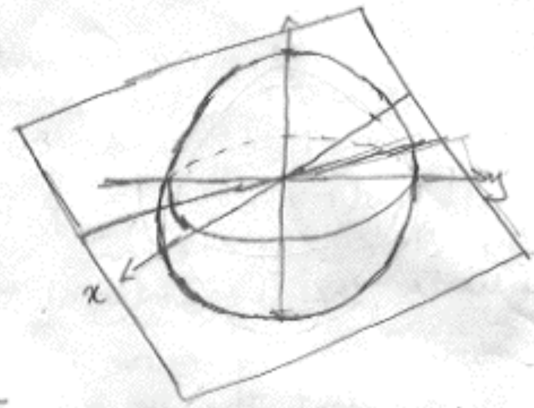
Question: Use Stokes' theorem to show that $\oint_C [y\,dx + z\,dy + x\,dz] = \sqrt{3}\,\pi a^2$ where C is the suitably oriented intersection of the surfaces $x^2+y^2+z^2=a^2$ and $x+y+z=0$

Solution:

Let S be any surface bounded by C.

Rewrite the line integral as $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$.

Then by Stokes theorem, $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot d\vec{S}$ where the direction of $d\vec{S}$ depends on the direction of C



Let's pick S to be the part of the plane $x+y+z=0$ lying inside C (any other surface bounded by C would work also)

Now evaluate the surface integral:

1) Sketch the region: see above

2) Pick a coordinate system
The curve of intersection is disk-shaped, and lies on a plane $z=-x-y$ ... let's try cylindrical.

3) Parametrize the surface
We know $z = -x-y = -r(\cos\theta + \sin\theta)$
Projecting onto the xy plane shows that $0\le r\le a$, $0\le\theta\le 2\pi$
$\therefore \vec{r}_S(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} - r(\cos\theta+\sin\theta)\hat{k}$

4) Integrand $= \text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \hat{\imath}[0-1] - \hat{\jmath}[1-0] + \hat{k}[0-1]$
$= -\hat{\imath} - \hat{\jmath} - \hat{k}$

5) Find $d\vec{S}$: a unit normal to the plane is $\hat{N} = \left(\frac{1}{\sqrt{3}}\right)[\hat{\imath}+\hat{\jmath}+\hat{k}]$
Also $dS = r\,dr\,d\theta$ (cylindrical)
$\therefore d\vec{S} = \pm(r/\sqrt{3})(\hat{\imath}+\hat{\jmath}+\hat{k})\,dr\,d\theta$ (orientation isn't specified in this problem; we'll pick the right sign at the end)

6) $\iint_S (\text{curl}\,\vec{F})\cdot d\vec{S} = \int_0^a dr \int_0^{2\pi} d\theta\, [\pm(r/\sqrt{3})(1+1+1)]$
$= \pm(3/\sqrt{3})\int_0^a r\,dr \int_0^{2\pi} d\theta = \pm\sqrt{3}\cdot 2\pi\cdot\left[r^2/2\right]_0^a = \pm\sqrt{3}\,\pi a^2$

So we pick the + sign, i.e. upward-pointing $d\vec{S}$ to get the right answer (this means the curve of intersection C was oriented clockwise looking from the top)

$\therefore \oint_C [y\,dx + z\,dy + x\,dz] = \sqrt{3}\,\pi a^2$ if C is oriented CW QED

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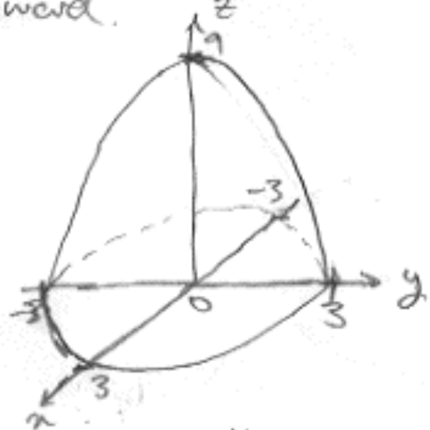
TUTORIAL 8 SOLUTIONS (NOVEMBER 2, 2006) – VERSION 1

Problem 7, Section 16.5 (Adams)

Question: Find the circulation of $\vec{F} = -y\hat{\imath} + x^2\hat{\jmath} + z\hat{k}$ around the oriented boundary of the part of the paraboloid $z = 9 - x^2 - y^2$ lying above the xy plane and having a normal field pointing upward.



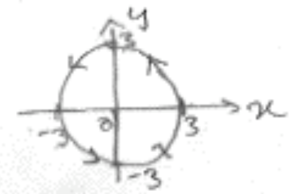
Solution:
we want $\oint_C \vec{F}\cdot d\vec{r}$; by Stokes theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}(\vec{F}))\cdot d\vec{S}$$

where, as the question indicates, $d\vec{S}$ points upward, and where S is any surface bounded by C.

Now the boundary in this case is simply the circle $x^2+y^2=9$, oriented CCW looking from the top (by the right-hand rule)

So take S to be the disk $x^2+y^2 \le 9$



1) Sketch the surface: see above
2) Pick a coordinate system: plane polar
3) Parametrize surface: $\vec{r}_S = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$, $(0 \le r \le 3) \wedge (0 \le \theta \le 2\pi)$
4) Integrand: $\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x^2 & z \end{vmatrix} = \hat{\imath}[0] - \hat{\jmath}[0] + \hat{k}[2x+1]$
$= (2x+1)\hat{k} = (2r\cos\theta + 1)\hat{k}$

5) $\hat{N} = +\hat{k}$, and $dS = r\,dr\,d\theta$, so $d\vec{S} = (r\hat{k})\,dr\,d\theta$
6) Evaluate the integral

$$\iint_S (\operatorname{curl}(\vec{F}))\cdot d\vec{S} = \int_0^3 dr \int_0^{2\pi} d\theta\,[2r^2\cos\theta + r]$$
$$= 2\int_0^3 r^2\,dr \int_0^{2\pi}\cos\theta\,d\theta + \int_0^3 r\,dr\int_0^{2\pi} d\theta \qquad \left(\text{Note: } \int_0^{2\pi}\cos\theta\,d\theta = 0\right)$$
$$= 0 + \left[\frac{r^2}{2}\right]_0^3 \cdot 2\pi = \pi\cdot 3^2 = 9\pi$$

$\therefore$ the circulation of $\vec{F}$ is $\boxed{9\pi}$